Modelling time and recursion

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Summary

1. Modelling time: clocks vs registers.
3. Solution technique: reduction to 1-BVASS(±).
Clocks vs registers

In a nutshell:
- Clocks record the *difference* between events’ timestamps.
- Registers record the events’ timestamps themselves.

The two approaches are essentially equivalent*.

*with uninitialised clocks (preserves emptiness)
Clocks vs registers

Consider the timed language over \{a, b\}^* “wait 1 sec after a and 2 sec after b”
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**CLOCKS**

- \( a, x = 1, x := 0 \)
- \( b, x = 2, x := 0 \)
- \( a, x = 2, x := 0 \)

**REGISTERS**

- \( a, x := 0 \)
- \( b, x := 0 \)
Consider the timed language over \{a, b\}^* “wait 1 sec after a and 2 sec after b”

\[
L = \{ (c_0,t_0), \ldots, (c_n,t_n) | (c_i = a \Rightarrow t_{i+1} - t_i = 1) \wedge (c_i = b \Rightarrow t_{i+1} - t_i = 2) \} 
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\begin{align*}
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\end{align*}
\]

**REGISTERS**

\[
\begin{align*}
\text{a, } t - x &= 1, \ x' = t \\
\text{b, } t - x &= 2, \ x' = t \\
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\text{b, } t - x &= 1, \ x' = t \\
\text{a, } t - x &= 2, \ x' = t
\end{align*}
\]

\( t: \) current timestamp

\( x: \) current

\( x': \) new
Clocks vs registers

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L = \{ (c_0, t_0), \ldots, (c_n, t_n) \mid (c_i = a \Rightarrow t_{i+1} - t_i = 1) \land (c_i = b \Rightarrow t_{i+1} - t_i = 2) \}
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Guard → Constraint

x: current
x’: new
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**guards → constraints**

**reset → constraint**
Timed register automata

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- \((\mathbb{N}, =)\): register automata [Kaminski, Francez TCS’94].
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- \((\mathbb{N}, =)\): register automata [Kaminski, Francez TCS’94].
- \((\mathbb{Z}, \leq, +1)\): discrete timed automata.
- \((\mathbb{R}, \leq, +1), (\mathbb{Q}, \leq, +1)\): dense timed automata.
Timed register automata

Fix finitely many registers $X = \{x, y, \ldots\}$. A timed register automaton is a tuple

$$A = \langle Q, I, F, \Delta, \varphi(\delta) \rangle$$

where

- $Q$ is a finite set of control states, with $I, F \subseteq Q$ the initial, final ones, resp.
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation.
- For every $\delta \in \Delta$, $\varphi(\delta)$ is a constraint using registers $X \cup X' \cup \{t\}$ over $(\mathbb{Q}, \leq, +1)$. 
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  - Atomic statements of the form: $x + 3 \leq y + 2$ with $x, y \in X \cup X' \cup \{t\}$. 
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  - Atomic statements of the form: $x + 3 \leq y + 2$ with $x, y \in X \cup X' \cup \{t\}$.
  - Boils down to conjunctions of $y - x \in I$, with $I$ an interval in $\mathbb{Q} \cup \{+\infty, -\infty\}$. 
This model is too powerful. Can simulate 2-counter machines.
Timed register automata vs Minsky

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**Minsky**

Let $c, d$ be two counters.

Basic operations:
- $c++$.
- $c--$.
- $c == 0$. 
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Let $x$, $y$, $z$ be three registers over $(\mathbb{N}, \leq,+1)$.

Transformation: $c \rightarrow x - z$, $d \rightarrow y - z$
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- Valuations $\rho : X \mapsto \mathbb{Q}$ are restricted to satisfy: $\max |\rho(x) - \rho(y)| \leq K$.
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Monotonic time *can be enforced* within the model:

- Add an extra register \( z \).
- Add to every transition the constraint

\[ z \leq t \land z' = t \]
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Check that the next timestamp is non-decreasing.

Save the timestamp.
Pushdown automata + time

TA + (untimed) stack

Pushdown timed automata
[Bouajjani, Echahed, Robbana HS’94]
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clocks on the stack are “frozen”
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- TA + timed stack
  - Dense-timed pushdown automata [Abdulla, Atig, Stenman LICS’12]

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TA + (untimed) stack \[\text{expressively equivalent}\] TA + timed stack

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TA + (untimed) stack expressively equivalent TA + timed stack

PUSHDOWN TIMED AUTOMATA
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DENSE-TIMED PUSHDOWN AUTOMATA
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RECURSIVE TIMED AUTOMATA
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TIMED REGISTER PUSHDOWN AUTOMATA
[C, Lasota LICS’15; C, Lasota, Lazić, Mazowiecki LICS’17]

clocks on the stack are “frozen” registers on the stack
In dtPDA [Abdulla, Atig, Stenman LICS’12]:

- Guards are of the form \( x \in I \).
- Clocks can be pushed on the stack (w.l.o.g. initialised to zero).
- Clocks on the stack evolve at the same rate as control clocks.
- Clock \( x \) can be popped from the stack if it satisfies the *pop guard* \( x \in I \).
Dense-timed pushdown automata

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Limitations:
- No diagonal control-control clock constraints (this is not a limitation).
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- Clock $x$ can be popped from the stack if it satisfies the \textit{pop guard} $x \in I$.

Limitations:
- No diagonal control-control clock constraints (this is not a limitation).
- No diagonal control-stack push clock constraints (unknown).
- No diagonal control-stack pop clock constraints (this is not a limitation).
Semantic collapse of dtPDA

Very strong collapse result:

**Theorem [CL LICS’15].** dtPDA of [AAS LICS’12] recognise the same class of timed languages as pushdown timed automata of [BER HS’94].
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In other words, the stack can be *untimed*.

Intuition:
- Time is monotone + stack LIFO policy
  \[\Rightarrow\] it suffices to keep track of finitely many pop constraints in the state
  \[\Rightarrow\] pop guards can be eliminated while preserving the timed language
Semantic collapse of dtPDA

Pop guards of the form $x \in [2, 3] + \text{time is monotone} + \text{stack LIFO policy}$

Upper bound constraints:

old subsumes new
Semantic collapse of dtPDA

Pop guards of the form $x \in [2, 3] + \text{time is monotone + stack LIFO policy}$

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$\leq 3$
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*old subsumes new*

$\leq 3$

- push(0)
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\[
\begin{align*}
\text{push(0)} & \quad x := 0 \\
\text{push(0)} & \quad \text{pop(} \leq 3) \\
\text{pop(} \leq 3) & \quad x \leq 3
\end{align*}
\]
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Pop guards of the form $x \in [2, 3] + \text{time is monotone} + \text{stack LIFO policy}$

Upper bound constraints: 

- $\text{old subsumes new}$

Lower bound constraints:

- $\text{new subsumes old}$

- $\text{new clock}$
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Consequences:
- dtPDA are expressively equivalent to TA + untimed stack (PDTA).
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Complexity:
- Add linearly many clocks and exponentially many control locations.
- Emptiness of PDTA is exponential in the number of clocks and polynomial in the number of control locations ⇒ emptiness of dtPDA is in EXPTIME.
Follow-ups to dtPDA

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Uezato, Minamide, “Synchronized Recursive Timed Automata”, LPAR’15. (diagonal + fractional constraints)

Li, Cai, Ogawa, Yuen, “Nested timed automata”, FORMATS’13. (reduction to dtPDA)

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Example from [BDKPT LATA’16] about logical characterisation of dtVPA.

\[ L = \text{words of the form } a^n b c^n d \text{ with } n \geq 0 \text{ s.t.} \]

- first \(c\) comes after 1 time unit after last \(a\)
- first \(a\) and last \(c\) are 2 time units apart
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We do not need a timed stack to recognise this language (4 clocks suffice). In fact, they show that the stack can be untimed in the spirit of [CL LICS’15].
Example

Consider the language of *timed palindromes* 

\[ L = \{ w w^R \mid w \in (\Sigma \times \mathbb{Q})^* \} \]

- It requires a truly timed stack.
- Cannot be expressed in any of the previous models.
Example

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- It requires a truly timed stack.
- Cannot be expressed in any of the previous models.
- It is a non-monotone language.
  - Can be made monotone by requiring palindromicity only for the fractional values.
Timed register pushdown automata

Fix a finite set of registers $X, Y$, input alphabet $\Sigma$, and stack alphabet $\Gamma$.  
A *timed register pushdown automaton* ($\text{trPDA}$) is a tuple

$$A = \langle Q, I, F, \text{PUSH}, \text{POP}, \{ \varphi(\delta) \mid \delta \in \text{PUSH} \cup \text{POP} \}, K \rangle$$

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- \( K \subseteq \mathbb{N} \) is a *boundedness threshold* for state and stack registers.
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Timed palindromes over $\Sigma = \{a, b\}$:

$L = \{ w w^R \mid w \in (\Sigma \times \mathbb{Q})^* \}$.

(a, t), push(a, y), y = t

(b, t), push(b, y), y = t
Timed palindromes over $\Sigma = \{a, b\}$: $L = \{ w \, w^R \mid w \in (\Sigma \times \mathbb{Q})^* \}$. 

Diagram:

- From p to q: $(a, t), \text{push}(a, y), y = t$ 
- From q to p: $(a, t), \text{pop}(a, y), y = t$ 
- From p to p: $(b, t), \text{push}(b, y), y = t$ 
- From q to q: $(b, t), \text{pop}(b, y), y = t$
Timed palindromes over $\Sigma = \{a, b\}$: 

$L = \{ w w^R \mid w \in (\Sigma \times \mathbb{Q})^* \}$.

The untiming projection of $L$ is a context-free language.
Example (2)

Untimed palindromes with the same number of a’s and b’s.

not a context-free language
Untimed palindromes with the same number of a’s and b’s.

\[ \varepsilon, \text{push}(\bot, y), y = x \]

\textit{not} a context-free language
Untimed palindromes *with the same number of a’s and b’s.*

Not a context-free language
Example (2)

Untimed palindromes with *the same number of a’s and b’s*. 

\[ \varepsilon, \text{push}(\perp, y), y = x \]

\[ \text{a, push(a), } x' = x + 1 \]

\[ \text{b, push(b), } x' = x - 1 \]

*not* a context-free language
Example (2)

Untimed palindromes with the same number of $a$’s and $b$’s.

$\varepsilon, \text{push}(\perp, y), y = x$

$a, \text{push}(a), x' = x + 1$

$\varepsilon, \text{nop}, x' = x$

$b, \text{push}(b), x' = x - 1$

$b, \text{pop}(b), x' = x$

Not a context-free language
Example (2)

Untimed palindromes with the same number of a’s and b’s.

not a context-free language
Deciding reachability

Timed automata

PSPACE
## Deciding reachability

<table>
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Deciding reachability

- Timed automata
- Pushdown timed automata (untimed stack)
- Timed register pushdown automata + monotone time (timed stack)
- Timed register pushdown automata (timed stack)

[AD TCS'94]

Word automaton
- LOGSPACE
- EXPONENTIAL

- EXPTIME
- NEXPTIME
- 2EXPTIME
- PSPACE
- 2EXPTIME
Deciding reachability

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[AD TCS'94]
Deciding reachability

- Timed automata
- Pushdown timed automata (untimed stack)
- Timed register pushdown automata + monotone time (timed stack)
- Timed register pushdown automata (timed stack)

- Word automaton
- Pushdown automaton
- Tree automaton

- Regions

[AD TCS'94]
[AGK CONCUR'16]

- Logspace
- P
- Exponential
- NEXPTIME
- 2EXPTIME
- PSPACE
- EXPTIME
Deciding reachability

Timed automata

Pushdown timed automata
(timed stack)

Pushdown timed automata
(untimed stack)

Timed register pushdown automata
+ monotone time
(timed stack)

Timed register pushdown automata
(timed stack)

[AD TCS'94]

regions

[AGK CONCUR'16]

[CL, LICS'15]

1-BVASS(\mathbb{Z}, =0)
(~1 \mathbb{Z}-counter tree automaton)

LOGSPACE PSPACE

P

EXPTIME

NP

NEXPTIME

2EXPTIME
# Deciding reachability

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<td>[CLLM, LICS’17]</td>
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- **1-BVASS**: 1-Bounded Vector Addition System
- **1-**
- **Z**: Integers
- **NP**: Nondeterministic Polynomial Time
- **PSPACE**: Polynomial Space
- **EXPTIME**: Exponential Time
- **LOGSPACE**: Logarithmic Space
- **NEXPTIME**: Nondeterministic Exponential Time
Conclusions

To model time + recursion:

- Registers are seemingly more powerful than clocks.
- We get an expressive model with decidable non-emptiness (2EXPTIME).
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- Registers are seemingly more powerful than clocks.
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Related models (not shown):
- Timed register context-free grammars (EXPTIME-c).
Conclusions

To model time + recursion:
- Registers are seemingly more powerful than clocks.
- We get an expressive model with decidable non-emptiness (2EXPTIME).

Related models (not shown):
- Timed register context-free grammars (EXPTIME-c).

Open questions:
- We have only an EXPTIME lower bound for our trPDA model.
- $1$-BVASS($\mathbb{Z}$, $\geq 0$, $\leq 0$) are in EXPTIME and PSPACE-hard.
- Truly expressive timed pushdown automata with clocks?