The Covering And Boundedness Problems For Vector Addition Systems
Charles Rackoff [1]
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Petri Nets and VAS

- **Petri Nets** are used to model concurrent systems
- Now, we look at Vector Addition Systems (VAS) which are equivalent to Petri Nets
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In our lectures, we have seen these problems are bounded by non-primitive recursive functions (Ackermann)
Petri Nets and VAS

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We study the complexity of the following problems:

- Coverability
- Boundedness

In our lectures, we have seen these problems are bounded by non-primitive recursive functions (Ackermann)
Now, we show they lie in **EXPSPACE**.
Definition: VAS

A Vector Addition System (VAS), of dimension $k$ is defined as a tuple $(v, A)$, where:

- $v \in \mathbb{Z}^k$ is the start vector
- $A \subseteq \mathbb{Z}^k$ is a finite set called the addition set
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**Paths in VAS**

A sequence of vectors $w_1, w_2, \ldots, w_m \in \mathbb{Z}^k$ is called a run of the system $(v, A)$, of length $m$ if

- $w_1 = v$
- $w_{i+1} - w_i \in A$, $\forall i \in \{1, 2, \ldots, m - 1\}$
Definitions

Reachability

A vector $v_1 \in \mathbb{Z}^k$ is reachable in $(v, A)$ if there exist a run $w_1, w_2, \ldots, w_m$ of vector addition system where

- $w_m = v_1$
- $w_i \in \mathbb{N}^k \quad \forall i \in \{1, 2, \ldots, m\}$
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Reachability and Coverability set

- $R(v, A) = \{w \in \mathbb{N}^k : w \text{ is reachable in } (v, A)\}$ is called the reachability set of $(v, A)$
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Reachability and Coverability set

- $R(v, A) = \{ w \in \mathbb{N}^k : w \text{ is reachable in } (v, A) \}$ is called the reachability set of $(v, A)$
- Similarly, $C(v, A) = \{ w \in \mathbb{N}^k : \exists v_1 \in R(v, A), w \leq v_1 \}$ is called the coverability set of $(v, A)$
How To Define Sizes

The size $n$ of a vector ($v \in \mathbb{Z}^k$) is defined as the sum of the lengths of the binary representations of the components (where the size of 0 is 1).
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### Definitions

#### How To Define Sizes

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The size $n$ of a VAS ($v, A$) is the size of $A$.

For the coverability problem, we define $n$ to be the maximum of the sizes of $A$ and $v_1$ (the vector to cover).
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Main Result Of This Paper

Coverability and boundedness problems are decidable in space $2^{cn \log n}$, where $n$ is the size of the problem and $c$ is some constant.
Definitions

**$i$-positive and $i\|r$-bounded**

- A vector $v$ is said to be $i$-positive if $v(j) \geq 0$ for $1 \leq j \leq i$

or

- A vector $v$ is said to be $i\|r$-bounded for some $r \in \mathbb{N}^{+}$ if $0 \leq v(j) < r$ for $1 \leq j \leq i$
Definitions

**i-positive and i∥r-bounded**

- A vector $v$ is said to be $i$-positive if $v(j) \geq 0$ for $1 \leq j \leq i$
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*i*-positive and *i||r*-bounded

- A vector $v$ is said to be *i*-positive if $v(j) \geq 0$ for $1 \leq j \leq i$
- A vector $v$ is *i||r*-bounded for some $r \in \mathbb{N}^+$ if $0 \leq v(j) < r$ for $1 \leq j \leq i$
- A sequence $p$ is *i*-positive (*i||r*-bounded) if every member of $p$ is *i*-positive (*i||r*-bounded)
Proof of Coverability

Coverability problem

The Coverability problem for vector addition system is to determine for a given VAS $(v, A)$ and a vector $v_1$, whether $v_1 \in C(v, A)$.

This is the same as saying that there is a $k$-positive, $k$-covering path in $(v, A)$, where a path $p = w_1, w_2, \ldots, w_m$ is $i$-covering if and only if $w_m(j) \geq v_1(j)$ for $1 \leq j \leq i$.

Bounding function

We will try to bound the length of the run of the system by introducing a function $f: \mathbb{N} \rightarrow \mathbb{N}$ which is defined as $f(i) = \max(m(i, v) : v \in \mathbb{Z}^k)$ where $m(i, v)$ is the length of the shortest $i$-positive, $i$-covering path in $(v, A)$.

What we will be interested in is an upper bound on the value of $f(k)$, where $k$ is the dimension of the vector addition system.
Proof of Coverability

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The Coverability problem for vector addition system is to determine for a given VAS \((v, A)\) and a vector \(v_1\), whether \(v_1 \in C(v, A)\) This is the same as saying that there is a \(k\)-positive, \(k\)-covering path in \((v, A)\), where a path \(p = w_1, w_2, \ldots, w_m\) is \(i\)-covering if and only if \(w_m(j) \geq v_1(j)\) for \(1 \leq j \leq i\)

Bounding function

- We will try to bound the length of the run of the system by introducing a function \(f : \mathbb{N} \to \mathbb{N}\) which is defined as \(f(i) = \max(m(i, v) : v \in \mathbb{Z}^k)\) where \(m(i, v)\) is the length of shortest \(i\)-positive, \(i\)-covering path in \((v, A)\)
- What we will be interested in is an upper bound on the value of \(f(k)\), where \(k\) is the dimension of the vector addition system
Proof of Coverability

**Lemma**

\[ f(0) = 1 \]
Proof of Coverability

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Proof.
Consider a vector \( w \in \mathbb{Z}^k \).
It is a 0-positive and 0-covering path in \((v, A)\).
Hence \( m(0, w) = 1 \) giving \( f(0) = 1 \).
Proof of Coverability

**Lemma**

\[ f(i + 1) \leq (2^n f(i))^{i+1} + f(i) \text{ for } 0 \leq i < k \]

**Proof.**

Let \( v \in \mathbb{Z}^k, 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive \((i + 1)\)-covering path in \((v, A)\). We make 2 cases:

- There exists an \((i + 1)\)\|(2^n f(i))\)-bounded \((i + 1)\)-covering path in \((v, A)\)
Proof of Coverability

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Let \( v \in \mathbb{Z}^k, 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive \((i + 1)\)-covering path in \((v, A)\). We make 2 cases:

- There exists an \((i + 1)|| (2^n f(i))\)-bounded \((i + 1)\)-covering path in \((v, A)\)
  - Then there must exist an \((i + 1)|| (2^n f(i))\)-bounded \((i + 1)\)-covering path in \((v, A)\) in which no two vectors agree on all of the first \(i + 1\) places
  - The length of such a sequence is \( \leq (2^n f(i))^{i+1} \)
Proof of Coverability

Lemma

\[ f(i + 1) \leq (2^n f(i))^{i+1} + f(i) \text{ for } 0 \leq i < k \]

Proof.

Let \( \nu \in \mathbb{Z}^k \), \( 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive \((i + 1)\)-covering path in \((\nu, A)\). We make 2 cases:

- Otherwise.
  
  There exists an \((i + 1)\)-positive \((i + 1)\)-covering path in \((\nu, A)\) which is not \((i + 1)\)|(\(2^n f(i)\))-bounded
  
  Then there exists sequences \( p_1, p_2 \) such that \( p_1 p_2 \) an \((i + 1)\)-positive \((i + 1)\)-covering path in \((\nu, A)\), where \( p_1 \) is \((i + 1)\)|(\(2^n f(i)\))-bounded and \( p_2 \) begins with a vector \( w \) which is not \((i + 1)\)|(\(2^n f(i)\))-bounded
  
  Without loss of generality assume \( w(i + 1) \geq 2^n f(i) \)
  
  Since \( p_2 \) is \( i \)-positive, \( i \)-covering path in \((w, A)\) there exists a path \( p'_2 \) of length \( \leq f(i) \) in \((w, A)\) which is \( i \)-positive and \( i \)-covering
Proof of Coverability

**Lemma**

\[ f(i + 1) \leq (2^n f(i))^{i+1} + f(i) \text{ for } 0 \leq i < k \]

**Proof.**

Let \( v \in \mathbb{Z}^k, 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive \((i + 1)\)-covering path in \((v, A)\). We make 2 cases:

- Otherwise.
  - By definition of the size of a vector, all places in all the vectors in \( A \cup \{v_1\} \) have absolute value \( \leq 2^n \)
  - We have \( w(i + 1) \geq 2^n f(i) \) and \( p_2' \) is of length \( \leq f(i) \)
  - \( p_2' \) is thus \((i + 1)\)-positive and \((i + 1)\)-covering (WHY?)
Proof of Coverability

Lemma

\[ f(i + 1) \leq (2^n f(i))^{i+1} + f(i) \text{ for } 0 \leq i < k \]

Proof.

Let \( v \in \mathbb{Z}^k, 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive \((i + 1)\)-covering path in \((v, A)\). We make 2 cases:

- Hence \( p_1p_2 \) is an \((i + 1)\)-positive, \((i + 1)\)-covering path in \((v, A)\) of length \( \leq (2^n f(i))^{i+1} + f(i) \)
Proof of Coverability

Theorem

The covering problem can be decided in space $2^{cn\log n}$ for some constant $c$

Proof.

We have $f(0) = 1$ and $f(i + 1) \leq (2^n f(i))^{i+1} + f(i)$ for $0 \leq i < k$
Proof of Coverability

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Proof.

We have $f(0) = 1$ and $f(i + 1) \leq (2^n f(i))^{i+1} + f(i)$ for $0 \leq i < k$.

Consider $g$ such that $g(0) = 2^{3n}$ and $g(i + 1) = (g(i))^{3n}$ for $0 \leq i < k$. 

Proof of Coverability

**Theorem**

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Consider $g$ such that $g(0) = 2^{3n}$ and $g(i + 1) = (g(i))^{3n}$ for $0 \leq i < k$.

It is easy to see $f(i) \leq g(i)$ for $0 \leq i \leq k$. (HOW?)
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It is easy to see $f(i) \leq g(i)$ for $0 \leq i \leq k$.

Hence $f(k) \leq g(k) \leq 2^{(3n)^k} \leq 2^{(3n)^n} \leq 2^{2c_1 n \log n}$ for some constant $c_1$. 


Proof of Coverability

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We have $f(0) = 1$ and $f(i + 1) \leq (2^n f(i))^{i+1} + f(i)$ for $0 \leq i < k$

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Hence $f(k) \leq g(k) \leq 2^{(3n)^k} \leq 2^{(3n)^n} \leq 2^{2c_1 n\log n}$ for some constant $c_1$

Hence $v_1 \in C(v, A)$ if and only if there exists a $k$-positive $k$-covering path in $(v, A)$ of length $\leq 2^{2c_1 n\log n}$ (where $c_1$ is independent of $n$)
Proof of Coverability

**Theorem**

The covering problem can be decided in space $2^{cn\log n}$ for some constant $c$

**Proof.**

Hence $v_1 \in C(v, A)$ if and only if there exists a $k$-positive $k$-covering path in $(v, A)$ of length $\leq 2^{2^c_1 n\log n}$ (where $c_1$ is independent of $n$)

The size of any vector on such a path is $\leq 2^{dn\log n}$ for some constant $d$ (WHY?)
Proof of Coverability

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The covering problem can be decided in space $2^{cn\log n}$ for some constant $c$.

Proof.

Hence $v_1 \in C(v, A)$ if and only if there exists a $k$-positive $k$-covering path in $(v, A)$ of length $\leq 2^{2c_1n\log n}$ (where $c_1$ is independent of $n$).

The size of any vector on such a path is $\leq 2^{dn\log n}$ for some constant $d$.

There is a non deterministic procedure which guesses a $k$-positive $k$-covering path and accepts if it finds one.

This operates in space $2^{dn\log n}$. 
Proof of Coverability

**Theorem**

The covering problem can be decided in space $2^{cn \log n}$ for some constant $c$.

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Hence $v_1 \in C(v, A)$ if and only if there exists a $k$-positive $k$-covering path in $(v, A)$ of length $\leq 2^{2c_1 n \log n}$ (where $c_1$ is independent of $n$).

The size of any vector on such a path is $\leq 2^{d n \log n}$ for some constant $d$.

There is a non determinsitic procedure which guesses a $k$-positive $k$-covering path and accepts if it finds one.

This operates in space $2^{d n \log n}$.

By **Savitch’s Theorem** [2], there is a determinsitic procedure for the covering problem which operates in space $2^{cn \log n}$ for some constant $c$. 

\[\square\]
Proof of Boundedness

Self Covering Paths

Let \( k \in \mathbb{N}^+ \) and let \( p = w_1, w_2, w_3, \ldots, w_m \) be a sequence of vectors in \( \mathbb{Z}^k, \ m > 1 \). Then \( p \) is said to be **self-covering** if \( w_j < w_m \) for some \( j, \ 1 \leq j \leq m \)
Proof of Boundedness

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Boundedness problem

The Boundedness problem for vector addition system is to determine for a given VAS \((\nu, A)\) if \( R(\nu, A) \) is finite, where \( R(\nu, A) \) denotes the reachability set.

\( R(\nu, A) \) is infinite if and only if there is a \( k \)-positive, self-covering path in \((\nu, A)\), where \( k \) is the dimension of the VAS.
Proof of Boundedness

Self Covering Paths

Let $k \in \mathbb{N}^+$ and let $p = w_1, w_2, w_3, \ldots, w_m$ be a sequence of vectors in $\mathbb{Z}^k$, $m > 1$. Then $p$ is said to be self-covering if $w_j < w_m$ for some $j$, $1 \leq j \leq m$.

Bounding function

- We will try to bound the length of the run of the system by introducing a function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is defined as $f(i) = \max(m(i, v) : v \in \mathbb{Z}^k)$ where $m(i, v)$ is the length of shortest $i$-positive, self-covering path in $(v, A)$.
- What we will be interested in is an upper bound on the value of $f(k)$, where $k$ is the dimension of the vector addition system.
Lemma

Let $0 \leq i \leq k$, $v \in \mathbb{Z}^k$, $r > 1$ such that there is an $i\|r$ bounded, self-covering path in $(v, A)$. Then there is an $i\|r$ bounded, self-covering path in $(v, A)$ of length $\leq r^{n\alpha}$ for some constant $\alpha$ independent of $n, v, r$.
Proof of Boundedness

**Lemma**

Let $0 \leq i \leq k$, $v \in \mathbb{Z}^k$, $r > 1$ such that there is an $i \parallel r$ bounded, self-covering path in $(v, A)$. There is an $i \parallel r$ bounded, self-covering path in $(v, A)$ of length $\leq r^{n\alpha}$ for some constant $\alpha$ independent of $n, v, r$.

**Lemma**

$f(0) \leq 2^{n\alpha}$ for constant $\alpha$ in above Lemma.
Proof of Boundedness

Lemma

Let $0 \leq i \leq k$, $v \in \mathbb{Z}^k$, $r > 1$ such that there is an $i \| r$ bounded, self-covering path in $(v, A)$. There is an $i \| r$ bounded, self-covering path in $(v, A)$ of length $\leq r^{n^{\alpha}}$ for some constant $\alpha$ independent of $n, v, r$.

Lemma

$f(0) \leq 2^{n^{\alpha}}$ for constant $\alpha$ in above Lemma.

Proof.

Let $v \in \mathbb{Z}^k$, $0 \leq i < k$ be such that there is a self-covering path in $(v, A)$. This is trivially a $0\|2$-bounded, self-covering path. Hence proved from above lemma (HOW?)
Proof of Boundedness

Lemma

(Lemma 4.5 in the paper) Let $0 \leq i \leq k$, $v \in \mathbb{Z}^k$, $r > 1$ such that there is an $i \parallel r$-bounded, self-covering path in $(v, A)$. There there is an $i \parallel r$-bounded, self-covering path in $(v, A)$ of length $\leq r^{n\alpha}$ for some constant $\alpha$ independent of $n, v, r$.

Lemma

$f(i + 1) \leq (2^n f(i))^{n\alpha}$ for $0 \leq i < k$ and $\alpha$ as in above Lemma

Proof.

Let $v \in \mathbb{Z}^k$, $0 \leq i < k$ be such that there exists an $(i + 1)$-positive self-covering path in $(v, A)$. We make 2 cases:
Proof of Boundedness

Lemma

(Lemma 4.5 in the paper) Let $0 \leq i \leq k$, $v \in \mathbb{Z}^k$, $r > 1$ such that there is an $i\|r$-bounded, self-covering path in $(v, A)$. There there is an $i\|r$-bounded, self-covering path in $(v, A)$ of length $\leq r^{n\alpha}$ for some constant $\alpha$ independent of $n, v, r$.

Lemma

$f(i + 1) \leq (2^nf(i))^{n\alpha}$ for $0 \leq i < k$ and $\alpha$ as in above Lemma

Proof.

Let $v \in \mathbb{Z}^k$, $0 \leq i < k$ be such that there exists an $(i + 1)$-positive self-covering path in $(v, A)$. We make 2 cases:

- There exists an $(i + 1)\|\|2^n f(i)$-bounded, self-covering path in $(v, A)$. Then by the above Lemma, there is an $(i + 1)\|\|2^n f(i)$-bounded self-covering path in $(v, A)$ of length $\leq (2^nf(i))^{n\alpha}$
Proof of Boundedness

Lemma

\[ f(i + 1) \leq (2^n f(i))^n \alpha \]
for \(0 \leq i < k\) and \(\alpha\) as in above Lemma

Proof.

Let \(v \in \mathbb{Z}^k\), \(0 \leq i < k\) be such that there exists an \((i + 1)\)-positive self-covering path in \((v, A)\). We make 2 cases:

- Otherwise.
  - There exists an \((i + 1)\)-positive self-covering path in \((v, A)\) which is not \((i + 1)||2^n f(i))\)-bounded. Call it \(w_1, w_2, \ldots w_m\)
  - Let \(j, 1 \leq j \leq m\), be such that \(w_j < w_m\) and let \(l, 1 \leq l \leq m\), be the smallest number such \(w_l\) is not \((i + 1)||2^n f(i))\)-bounded
Proof of Boundedness

**Lemma**

\[ f(i + 1) \leq (2^n f(i))^\alpha \] for \( 0 \leq i < k \) and \( \alpha \) as in above Lemma

**Proof.**

Let \( v \in \mathbb{Z}^k, 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive self-covering path in \((v, A)\). We make 2 cases:

- **Otherwise.**
  - There exists an \((i + 1)\)-positive self-covering path in \((v, A)\) which is not \((i + 1)||\(2^n f(i))\)-bounded. Call it \( w_1, w_2, \ldots, w_m \)
  - Let \( j, 1 \leq j \leq m \), be such that \( w_j < w_m \) and let \( l, 1 \leq l \leq m \), be the smallest number such \( w_l \) is not \((i + 1)||\(2^n f(i))\)-bounded
  - Let \( v_1, v_2, \ldots, v_x \) be the shortest path in \((v, A)\) such that \( v_x \) agrees with \( w_l \) on the first \( i + 1 \) places, and the path \( p_0 = v_1, v_2, \ldots, v_{x-1} \) is \((i + 1)||\(2^n f(i))\)-bounded
  - No two elements in \( p_0 \) can agree on the first \( i + 1 \) places, or then the sequence could be made even shorter. Hence \( x - 1 \leq (2^n f(i))^{i+1} \)
Proof of Boundedness

**Lemma**

\[ f(i + 1) \leq (2^n f(i))^\alpha \text{ for } 0 \leq i < k \text{ and } \alpha \text{ as in above Lemma} \]

**Proof.**

Let \( v \in \mathbb{Z}^k, 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive self-covering path in \((v, A)\). We make 2 cases:

- **Otherwise.**
  - Without loss of generality assume \( v_x(i + 1) = w_l(i + 1) \geq 2^n f(i) \)
  - For \( 1 \leq u < m \), define \( a_u \in A \) by \( a_u = w_{u+1} - w_u \)
  - Consider the sequence \( a_l, a_{l+1}, \ldots a_{m-1}, a_j, a_{j+1}, \ldots, a_{m-1} \)
  - Let \( p_0 \) be the corresponding path obtained on firing these subsequently at \( v_x \). \( p_0 \) is \((i + 1)\)-positive (and hence \(i\)-positive) and self-covering (WHY?)
  - Let \( p \) be an \(i\)-positive and self-covering path in \((v_x, A)\) of length \( \leq f(i) \)
  - Since \( v_x(i + 1) \geq 2^n f(i) \) and each place of each vector in \( A \) has absolute value at most \(2^n\), \( p \) is in fact \(i + 1\) positive
Proof of Boundedness

Lemma

\[ f(i + 1) \leq (2^nf(i))^\alpha \] for \( 0 \leq i < k \) and \( \alpha \) as in above Lemma

Proof.

Let \( v \in \mathbb{Z}^k \), \( 0 \leq i < k \) be such that there exists an \((i + 1)\)-positive self-covering path in \((v, A)\). We make 2 cases:

- Hence \( p_0p \) is an \((i + 1)\)-positive, self-covering path in \((v, A)\) of length
  \[ \leq (2^nf(i))^{i+1} + f(i) \leq (2^nf(i))^\alpha \] (assuming \( \alpha > 1 \))
Proof of Boundedness

**Theorem**

The boundedness problem can be decided in space $2^{cn \log n}$ for some constant $c$

**Proof.**

We have $f(0) \leq 2^{n^\alpha}$ and $f(i + 1) \leq (2^n f(i))^{n^\alpha}$ for $0 \leq i < k$
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Using the same proof technique that we did in the coverability section we get that $f(k) \leq 2^{c_1 n \log n}$ for some constant $c_1$
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Hence $R(v, A)$ is infinite iff there exists a $k$-positive, self-covering path in $(v, A)$ of length $2^{2c_1 n \log n}$ (where $c_1$ is independent of $n$)
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Hence $R(v, A)$ is infinite iff there exists a $k$-positive, self-covering path in $(v, A)$ of length $2^{2c_1 n\log n}$ (where $c_1$ is independent of $n$)

The size of any vector on such a path is $\leq 2^{dn\log n}$ for some constant $d$
Proof of Boundedness

**Theorem**

The boundedness problem can be decided in space $2^{cn\log n}$ for some constant $c$

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We have $f(0) \leq 2^{n^\alpha}$ and $f(i + 1) \leq (2^n f(i))^{n^\alpha}$ for $0 \leq i < k$

Using the same proof technique that we did in the coverability section we get that $f(k) \leq 2^{2c_1 n\log n}$ for some constant $c_1$

Hence $R(v, A)$ is infinite iff there exists a $k$-positive, self-covering path in $(v, A)$ of length $2^{2c_1 n\log n}$ (where $c_1$ is independent of $n$)

The size of any vector on such a path is $\leq 2^{dn\log n}$ for some constant $d$

There is a non deterministic procedure which guesses a $k$-positive self-covering path and operates in space $2^{dn\log n}$
Proof of Boundedness

Theorem

The boundedness problem can be decided in space $2^{cn\log n}$ for some constant $c$

Proof.

We have $f(0) \leq 2^{n^\alpha}$ and $f(i + 1) \leq (2^n f(i))^{n^\alpha}$ for $0 \leq i < k$

Using the same proof technique that we did in the coverability section we get that $f(k) \leq 2^{c_1 n\log n}$ for some constant $c_1$

Hence $R(\nu, A)$ is infinite iff there exists a $k$-positive, self-covering path in $(\nu, A)$ of length $2^{c_1 n\log n}$ (where $c_1$ is independent of $n$)

The size of any vector on such a path is $\leq 2^{d n\log n}$ for some constant $d$

There is a non deterministic procedure which guesses a $k$-positive self-covering path and operates in space $2^{d n\log n}$

By **Savitch’s Theorem**, there is a deterministic procedure for boundedness problem which operates in space $2^{cn\log n}$ for some constant $c$
What We Have Shown

Coverability and boundedness problems are decidable in space $2^{cn\log n}$, where $n$ is the size of the problem and $c$ is some constant.

This is much better than the non-primitive recursive bounds seen in class. Also, this is very close to the known lower bounds, which are $2^{c\sqrt{n}}$ [R. Lipton, 1976] [3]
Further Reading

Further Extensions to Rackoff’s Technique

1. Rémi Bonnet and Alain Finkel and M. Praveen
   Extending the Rackoff technique to Affine nets
   FSTTCS 2012 [4]

2. Ranko Lazic and Sylvain Schmitz
   The Ideal View on Rackoff’s Coverability Technique
   Reachability Problems - 9th International Workshop 2015 [5]
Charles Rackoff.
The covering and boundedness problems for vector addition systems.

Walter J. Savitch.
Relationships between nondeterministic and deterministic tape complexities.

R. Lipton.
The reachability problem requires exponential space.
Rémi Bonnet, Alain Finkel, and M. Praveen. 
Extending the rackoff technique to affine nets. 

Ranko Lazic and Sylvain Schmitz. 
The ideal view on rackoff’s coverability technique. 