Boundedness of 2-Reset Petri Nets

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**ABSTRACT.** We study Petri nets with Reset arcs. These can be seen as counter machines with some restricted set of operations. It is known that boundedness is undecidable for nets with three or more reset arcs. In this report, we study the decidability of boundedness for nets with up to two Reset arcs.

1. **Introduction**

Petri Nets are a class of infinite state transition systems. Reset Nets are Petri Nets extended with the possibility of setting the content of places to zero in a single transition. Adding reset arcs to a Petri Net makes the net more expressive and pushes Petri nets closer to the frontiers of decidability.

It was shown in [2], that Boundedness is undecidable for Reset Nets. Further, in [3], it was shown that boundedness is undecidable for nets with 3 or more reset arcs. In this report, we study the result shown in Section 4 of [3] and complete the proof wherever necessary:

**INPUT:** A Reset net $N = \langle P, T, F, F_R \rangle$ with up to 2 reset arcs, and an initial marking $m_0$.

**QUESTION:** Is the reachability set of $N$ finite?

The assumptions, definitions and notations are kept consistent with that in [3] to make the proof easier to read and understand.

Finally in Section 3, we give a brief idea of the proof used in [1] to show that place-boundedness is decidable for the same.

2. **Main Proof**

In this section, we prove the main result, i.e. boundedness for Reset nets with at most two reset arcs is decidable.

First we observe that the boundedness problem is obviously semi-decidable (for any number of reset arcs) - the standard construction of the reachability tree finishes by enumerating all reachable markings. Therefore the semi-decidability of unboundedness is what should be demonstrated.

**Proof Sketch** (See figure 1): We start with an easy sub-case in Section 2.1, where we first introduce the notion of a *simple witness* and then try to find properties in a net that guarantee its existence. Then in Section 2.2, we introduce a *regular witness* and show that an unbounded net without a simple witness and at most two reset arcs must have a regular witness. In section 2.3, we prove an important lemma from Section 2.2, namely Lemma 15. The following are helpful:

**Definition 1.** Assuming a given initial marking $m_0$, an unbounded run is an infinite transition sequence $t_1, t_2, \ldots$ firable from $m_0$ (i.e. there is a sequence of markings $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} m_2 \ldots$), such that some place(s) are unbounded (get ever larger values) in the marking sequence.

The run is said to be non-decreasing if $\forall i, j : i < j \implies m_i \geq m_j$.

**Lemma 2.** (Lemma 1 in [3]) A reset net is unbounded iff it has an unbounded non-decreasing run.

**Proof.** The *if* direction is trivially true.

We only show the *only-if* direction. Whenever some $m$ can be reached from $m_0$, then some $m' \geq m$ can be reached from $m_0$ via a non-decreasing marking sequence – the sequence of transitions reaching $m'$ is obtained from the sequence reaching $m$ by removing all sub-sequences $\sigma$ where $m_i \xrightarrow{\sigma} m_j$ has $m_i \geq m_j$. We use the fact that reset nets exhibit monotonicity, i.e, $m_1 \xrightarrow{\sigma} m_2$ and $m_1' \geq m_1 \implies m_1' \xrightarrow{\sigma} m_2'$ with $m_2' \geq m_2$. Now if from $m_0$ the net can reach an infinite number of distinct markings, then we have an infinite number of (finite) non-decreasing marking sequences. We arrange all these in a tree. Since the set of transitions is finite, each node in the tree has a
finite number of children. Then we can use Koenig's Lemma to show the existence of an infinite non-decreasing run.

In the following sections we always implicitly assume that a net carries some fixed initial marking (usually denoted by $m_0$).

2.1. Simple witnesses.

**Definition 3.** (Definition 1 in [3]) For a marked Reset net, a simple witness is a 4-tuple $m_1, m_2, u, p$ such that $m_1$ is reachable, $m_1 \xrightarrow{u} m_2$, $m_1 \leq m_2$, $m_1(p) < m_2(p)$, and $p$ is not reset by any transition in the sequence $u$. Repeating $u$ ad infinitum (from $m_1$) yields a run $R$ along which $p$ is unbounded. Hence a simple witness surely witnesses unboundedness. Note that a simple witness (if any) is surely encountered and recognized when constructing the reachability tree. Therefore, unboundedness is semi-decidable for nets with simple witnesses. The next two Lemmas show when a simple witness exists in a net.

**Lemma 4.** (Lemma 7 part (a) in [3] - stronger) A marked reset net admits a simple witness iff it has a run along which a place is unbounded and reset only finitely many times.

**Proof.** ($\Rightarrow$) Let $m_1, m_2, u, p$ be the simple witness and $\sigma$ be such that $m_0 \xrightarrow{\sigma} m_1$ (exists as $m_1$ is reachable). Now consider the run $\sigma u^\omega$ (repeat $u$ to infinity after $\sigma$). Along this run the place $p$ is unbounded and reset only finitely many times (resets of $p$ may occur only in $\sigma$).

($\Leftarrow$) We have a run along which a place (say $p$) is unbounded and reset only finitely many times. In the run we go to the marking obtained after the last reset of place $p$. After this we must be able to find an infinite sub-sequence of markings in which the value at place $p$ is strictly increasing (since $p$ is unbounded). In this sub-sequence we can use Dickson’s Lemma to find two markings $m'_1$ and $m'_2$ such that $m'_1$ comes before $m'_2$ in the sub-sequence (i.e. $\exists u : m'_1 \xrightarrow{u} m'_2$) and $m'_1 < m'_2$. We also know that $m'_1(p) < m'_2(p)$, and hence we have found a simple witness.

**Lemma 5.** (Lemma 7 part (b) in [3]) If a marked reset has a run along which only one place is unbounded, then it admits a simple witness.

**Proof.** If along the run, the unbounded place (say $p$) is reset only finitely many times, we use the above Lemma to show the existence of a simple witness. So we consider the case when $p$ is reset infinitely many times along the run. We now fix a (large) integer $K$. Since place $p$ is unbounded along the run, there is a marking $m'$ such that $m'(p) \geq K$. We now go to the marking obtained after the last reset before $m'$ (call this marking $m''_0$). We always have a large enough $K$ so that
there is at least one reset before reaching \( m' \) (or we will have unboundedness without reset arcs - simple witness from above Lemma). Since \( m_0(p) = 0 \) and \( m'(p) \geq K \), we must be able to find a sub-sequence of markings \( m_0', m_1', \ldots, m' \) in which the value at place \( p \) is strictly increasing and no resets take place in this sub-sequence. The length of the sub-sequence must me at least \( K \) (in one transition we can increase the value at a place by at most a fixed constant, equal to one in our case). Since the only unbounded place is \( p \), the number of markings taken by the rest of the places is finite (say \( k \)). Thus if \( K \) is such that \( K > k \), using pigeon-hole principle we will be able to find two markings (say \( m_1' \) and \( m_2' \)) in this sub-sequence that have the same value at each place (except \( p \)) and \( m_1'(p) < m_2'(p) \). It is clear that we have found a simple witness. \( \square \)

From the above Lemmas, it is easily seen that unbounded nets with one reset arc must have a simple witness (we have a run along which either only the place with the reset is unbounded or we have a place without any resets that is unbounded).

Only nets with two reset arcs and without a simple witness now remain of interest, which we deal with in Section 2.2.

2.2. Regular witnesses.

**Convention 6.** In this section, we fix an unbounded net \( N \) without a simple witness, i.e., with two Reset arcs, each connected to a different place, where along every unbounded run precisely the two resetable places are unbounded and both are reset infinitely often. The two resettable places are denoted by \( p_1 \) and \( p_2 \). A marking in \( N \) is denoted by \( (r, x, y) \), where \( r \) ranges over the set of sub-markings of non-resettable places (denoted by \( fcs - a \) finite control state) and \( x, y \) denote the values at \( p_1, p_2 \) respectively. By \( (r, -, y) \) we denote that the value for \( p_1 \) is don’t care (similarly for \( (r, x, -) \) and \( p_2 \)).

By \( m \xrightarrow{u} \; m' \) (or \( m \xrightarrow{n} \; m' \)) we denote that \( u \) (or that \( \exists u : m \xrightarrow{u} m' \), which) contains at most \( n \) reset transitions.

**Lemma 7.** The set \( \{m' \mid m\xrightarrow{n} m' \} \) (and \( \{m' \mid m\xrightarrow{+} m' \} \) )is finite for any reachable \( m \) and any \( n \in \mathbb{N} \).

**Proof.** This is obvious since if unboundedness is encountered in a finite number of reset transitions, we will have a simple witness (using Lemma 4). \( \square \)

**Definition 8.** (Definition 2 in [3]) By a path scheme of order \( n \), related to \( r, r', b_1, b_2 \), and the pair \( \langle p_2, p_2 \rangle \), where \( r, r' \) are sub-markings on the non-resettable places and \( b_1, b_2 \in \mathbb{N} \), we mean a triple \( (w, f, x_0) \), where \( w : \mathbb{N} \to T^+, f : \mathbb{N} \to \mathbb{N} \), \( x_0 \in \mathbb{N} \) such that \( \forall x \geq x_0 : (r, x, b_1) \xrightarrow{w(x)} (r', f(x), b_2) \). Note that each \( w(x) \) has at most \( n \) reset transitions.

Besides the pair \( \langle p_2, p_2 \rangle \), we also consider any general pair \( \langle p, p' \rangle \) where \( p, p' \in \{p_1, p_2 \} \), meaning that \( b_1 \) is value for \( p \) and \( b_2 \) for \( p' \) (similarly for \( x \) and \( f(x) \) for the corresponding resettable places other than \( p \) and \( p' \) respectively).

The path scheme is said to have the maximum property (for the pair \( \langle p_2, p_2 \rangle \) ) if \( \forall x \geq x_0 : \exists y > f(x) \) such that \( (r, x, b_1) \xrightarrow{n} (r', y, b_2) \).

**Lemma 9.** Given a path scheme \( (w, f, x_0) \) related to \( r, r', b_1, b_2, \langle p, p \rangle \) (i.e., \( \langle p_1, p_1 \rangle \) or \( \langle p_2, p_2 \rangle \)), then \( x_0 \leq x \leq x' \implies f(x) \leq f(x') \).

**Proof.** We prove the Lemma for the pair \( \langle p_2, p_2 \rangle \). Let \( m_1 = (r, x, b_1) \), \( m_1' = (r, x', b_1) \) and \( m_2 = (r', f(x), b_2) \). Now \( x \leq x' \) implies \( m_1 \leq m_1' \) and by monotonicity we have \( m_1 \xrightarrow{w(x)} m_2 \) implies \( m_1' \xrightarrow{w(x)} m \), where \( m = (r', y, b_2) \) and \( m \geq m_2 \), which shows \( y \geq f(x) \). Now using the maximum property of the path scheme, we must have \( f(x') \geq y \geq f(x) \). \( \square \)

**Definition 10.** (In end of Lemma 8 in [3]) We call a collection \( r, r', b_1, b_2, \langle p, p' \rangle, n \) as sensible if there exists a path scheme of order \( n \) related to \( r, r', b_1, b_2, \langle p, p' \rangle \).
**Lemma 11.** Given a sensible collection $r, r', b_1, b_2, \langle p, p' \rangle, n$ there is a path scheme of order $n$, which is related to $r, r', b_1, b_2, \langle p, p' \rangle$ and has the maximum property.

**Proof.** We prove the Lemma for the pair $\langle p_2, p_2 \rangle$. Using Lemma 7, we have $\forall x \geq x_0$, the set $\{m = (r', -, b_2) : (r, x, b_1) \rightarrow_n^+ m\}$ is finite, and following Definition 10, we have that the set is non-empty, since a path scheme (not necessarily having the maximum property) exists. For each $x$ we take a marking $m$ such that the non-$b_2$ place has a maximal value and call the corresponding sequence of transitions $w(x)$.

The rest of the proof is structured as follows:

1. We define an increasing pair (Definition 12) and show its existence (Lemma 13).
2. We define a regular path scheme (Definition 14) and show its existence (Lemma 15).
3. We define a regular witness (Definition 16) and use the existence of an increasing pair and a regular path scheme to show its existence (Proposition 17). It is easy to see decidability using a regular witness and this will complete our proof.

**Definition 12.** Given an unbounded non-decreasing run $R$ in $N$, a pair $(r, n)$ in $R$ is called increasing if $\forall m \in \mathbb{N}$, there are $x_1 < x_2 < \cdots < x_{m+1}$ such that $(r, x_1, 0) \rightarrow_n^+ (r, x_2, 0) \rightarrow_n^+ \cdots \rightarrow_n^+ (r, x_{m+1}, 0)$ in $R$.

**Lemma 13.** Given an unbounded non-decreasing run $R$ in $N$, there is an increasing pair $(r, n)$ in $R$.

**Proof.** First we note that there cannot be more than $k = |\text{fcs}|$ resets of $p_2(p_1)$ without a reset of $p_1(p_2)$ in between or else using the pigeonhole principle we are able to find a simple witness. Thus bounding the number of resets of either $p_1$ or $p_2$ bounds the number of resets of other.

We use induction on the number of possible markings on non-resettable places ($= k$).

For $k = 1$ : we consider all the markings obtained after reset of place $p_2$. They are of the form $(r, x_1, 0), (r, x_2, 0), \ldots$ where we also have $x_1 < x_2 < \ldots$, since the sequence is non-decreasing. Also the number of resets between two consecutive resets of $p_2$ is bounded. Hence we have found an increasing pair.

Now we assume the statement is true for some $k \geq 1$, and prove it for $k + 1$. Let the markings be denoted by $r_1, r_2, \ldots, r_{k+1}$. We again consider all the resets of $p_2$. Among the sequence of markings obtained, look at all those with the non-resettable places having marking $r_{k+1}$. We have the following cases:

1. If the number of such markings is finite : we are able to find an increasing pair after these markings, using the inductive hypothesis.
2. If the number of resets between any two such consecutive markings is bounded (say by $n$) : we have an increasing pair, i.e. $(r_{k+1}, n)$.
3. In the case when the number of such markings is infinite and the number of resets between two consecutive markings is unbounded : $\forall m \in \mathbb{N}$, we are able to find two markings such that the number of resets between them is large enough to find $m$ terms using the inductive hypothesis on $k$ markings taken by the non-resettable places.

**Definition 14.** (Definition 3 and 4 in [3] combined) A path scheme $(w, f, x_0)$ is said to be regular if $f(x) = \rho x + h(x \text{ mod } d)$ for some $d \in \mathbb{N}$, some rational $\rho$ and some (finite) function $h$, and for each $i \in \{1, 2, \ldots, d - 1\}$, there are $m \in \mathbb{N}, u_1, v_1, u_2, v_2, \ldots, u_m, v_m, u_{m+1} \in T^*$ and functions $g_j : N \rightarrow N$ for $j \in \{1, 2, \ldots, m\}$, such that $g_j(x) = \rho_j x + b_j$ for some rational $\rho_j, b_j$ where the following holds for every $x \geq x_0, x \text{ mod } d = i : w(x) = u_1 v_1^{g_1(x)} u_2 v_2^{g_2(x)} \cdots u_m v_m^{g_m(x)} u_{m+1}$. Note that $\rho d$ and $\rho_j d$ must be integers.

**Lemma 15.** (Lemma 8 in [3]) Given a sensible collection $r, r', b_1, b_2, \langle p, p' \rangle, n$ there is a regular path scheme of order $n$, which is related to $r, r', b_1, b_2, \langle p, p' \rangle$ and has the maximum property.

**Proof.** In Section 2.3.
Definition 16. (Definition 5 in [3]) A regular witness (for the net $N$) is a reachable marking $(r, x_0, 0)$ together with a regular path scheme $(w, f, x_0)$ (of some order $n$) related to $r, r, 0, 0, (p_2, p_2)$, which has the property $\forall x \geq x_0 : x < f(x)$.

Observe that, for $f(x) = \rho x + h(x \mod d)$, there is $x_0$, such that $\forall x \geq x_0 : x < f(x)$ is true precisely when $\rho > 1$ or $\rho = 1$ and $h(i) > 0$ for all $0 \leq i < d$.

A regular witness surely encounters unboundedness. Moreover, such a witness, accompanied with a sequence demonstrating reachability of $(r, x_0, 0)$ can be represented as a string in a fixed alphabet. Whether or not a given string represents a regular witness can be checked easily. Therefore when a regular witness exists, it can be found by generating and checking all strings (brute force enumeration). Thus it remains to show that a regular witness is guaranteed to exist.

Proposition 17. (Proposition 1 in [3]) For an unbounded net with at most two Reset arcs and without a simple witness, there is a regular witness.

**Proof.** It suffices to show the claim for our fixed net $N$ and an unbounded non-decreasing run $R$ in it. Using Lemma 15, we are guaranteed the existence of a regular path scheme $(w, f, y_0)$ of order $n$ which is related to $r, r, 0, 0$ and has the maximum property (the collection is sensible since $(r, -, 0)$ appears infinitely often in $R$). Consider an increasing pair $(r, n)$ with $m > d$ and values $x_1, x_2, \ldots, x_{m+1}$ in the non-zero place. Also make $m$ large enough such that $y_0 + d \leq x_{m+1}$. We consider $d$ values of $x \geq y_0$, each with different $x \mod d$. Now for each of these $x$, there exists $x_i$, such that $x_i \leq x < x_{i+1}$ and using the maximum property of the path scheme along with Lemma 9, we get that $f(x) \geq f(x_i) \geq x_{i+1} > x$. From this we have $\rho > 1$ or $\rho = 1$ and $h(i) > 0$ for all $0 \leq i < d$ (we are dealing with large enough $x$ so that the finite function $h$ is small compared to $\rho x$). 

2.3. **Proof of Lemma 15 - Existence of a Regular Path Scheme.** (Lemma 8 in [3] - Non trivial lines of the proof are considered as Lemmas and proved separately)

We start with a lemma dealing with the composition of schemes. We implicitly suppose that relevant schemes are ‘composable’ in the natural sense.

**Lemma 18.** Composition of two regular path schemes is a regular path scheme.

**Proof.** Let the two path schemes be $(w_1, f_1, x_{01})$ and $(w_2, f_2, x_{02})$. We simply define the composition with $f(x) = f_2(f_1(x))$ and $w(x) = w_1(x)w_2(f_1(x))$ and suitable $x_0$.

Let $f_1(x) = \rho_1 x + h_1(x \mod d_1)$ and $f_2(x) = \rho_2 x + h_2(x \mod d_2)$. Consider $f(x) = f_2(f_1(x)) = \rho_2(\rho_1 x + h_1(x \mod d_1)) + h_2((\rho_1 x + h_1(x \mod d_1)) \mod d_2)$. Putting $\rho = \rho_1 \rho_2$, $d = d_1 d_2$, and observing $\rho_1 x$ can be replaced with $\rho_1(x \mod d_1 d_2)$ in the expression, it is clear that $f(x) = \rho x + h(x \mod d)$. 

*Figure 2.* (Fig. 1 in [3]) The idea behind Definition 14: The only kind of unbounded run in this net with $m_0 = (1, 0, 1, 0)$ is $\prod_{n=1}^{\infty} t_1^n t_2^m t_4^n t_4^m$. 

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It can be checked that \( w(x) \) follows the pattern required by a regular path scheme by similarly reasoning for \( w_1(x) \) and \( w_2(x) \).

A key point is to prove Lemma 15 for the case \( n = 0 \). In fact it suffices to suppose \( b_1 = 0 \) and \( b_2 = -1 \) (don’t care). Thus first we prove that if \( r,r',0,-,\langle p,p'\rangle,0 \) is a sensible collection then there is a regular path scheme of order \( 0 \) which is related to \( r,r',0,-,\langle p,p'\rangle \) and has the maximum property.

**Definition 19.** A sequence \((r_1,x_1,y_1),(r_2,x_2,y_2),\ldots,(r_m,x_m,y_m)\) with \( m \geq 2 \), \( r_1 = r_m \), (and no resets in between) is called a cycle. An elementary cycle is one which doesn’t contain any shorter cycle. We also attach the integers \( \Delta x = x_m - x_1 \) and \( \Delta y = y_m - y_1 \). It is clear that we must have \( \Delta y/\Delta x < 0 \) for the sequence to be non-decreasing and not have a simple witness.

Note that an elementary cycle can have length at most \( k = |fcs| \).

We note that for \( n = 0 \) we are considering the case with no resets (i.e. \(* \to_0 \)). We fix a sufficiently large constant \( b \in \mathbb{N} \), for example \( b = k! \) where \( k = |fcs| \) (recall that \( fcs \) is the set of sub-markings over the non-resettable places).

Let \((w,f,x_0)\) be a path scheme (not necessarily regular) of order \( 0 \) related to our sensible collection.

We take some \( x \geq x_0 \) and consider \( w(x) \) (with no reset transitions) and denote the corresponding marking sequence \( Seq \) as \((r_1,x_1,y_1),(r_2,x_2,y_2),\ldots,(r_m,x_m,y_m)\) with \( r_1 = r, r_m = r' \) and other consistency conditions. We can imagine the run as a tour in the coordinate plane with values of \( p_1,p_2 \) as coordinates and control state as additional (bounded) information.

**Convention 20.** We suppose:

1. \( Seq \) is non-decreasing (we can remove the segments between a pair of markings that is ‘decreasing’, and use monotonicity for further transitions).
2. \( w(x) \) is the shortest among all paths that give the maximal \( f(x) \).

**Lemma 21.** Only for finitely many \( j \), \( 1 \leq j \leq m \) (the bound not depending on \( x \)) we can have that \( x_j \leq b \) or \( y_j \leq b \), i.e. we are in the horizontal or vertical belt of breadth \( b \) in the plane.

**Proof.** Suppose that there are unboundedly many values in some part, say \( y \leq b \). We are able to find two markings \( m_i \) and \( m_j \) (with \( i < j \)) such that \( r_i = r_j \) and \( y_i = y_j \) (existence is guaranteed using the pigeon-hole principle since both \( r \) and \( y \) are bounded and the number of markings is unboundedly large). Now using the fact that \( Seq \) is non-decreasing we have \( x_i < x_j \). Hence we have found a simple witness, which is not possible.

**Definition 22.** Consider a sub-part of \( Seq \) \((r_1,x_1,y_1),(r_{l+1},x_{l+1},y_{l+1}),\ldots,(r_{l+n},x_{l+n},y_{l+n})\) (\( 1 \leq l \), \( 2 \leq n \), \( l + n \leq m \)) such that \( x_j > b \) and \( y_j > b \) for all \( j : l + 1 \leq j \leq l + n - 1 \). It is called a traversal if either \( x_l = b \) and \( x_{l+n} = b \) or \( y_l = y_{l+n} = b \). Graphically, a traversal is when we tour from one belt (horizontal or vertical, of breadth \( b \)) to another, and retour is when we come back to the same belt.

**Convention 23.** We can suppose that each traversal and retour finish with the maximal value on the plane with the value other than \( b \) (keeping the same \( r \) and \( r' \)). This is obvious since by monotonicity we can perform the same operations from the maximal state as from any other state.

Now we state Lemmas 24 - 27 whose proofs we give in the end.

**Lemma 24.** The final marking \((r',x_m,y_m)\) must have \( x_m \leq b \) or \( y_m \leq b \).

**Lemma 25.** For any bounded segment \((r,x',b_1) \to_0^\delta (r',f(x'),b_2)\) (and similarly for any general pair \( \langle p,p' \rangle \)) with large enough \( x' \), there is \( \delta \in \mathbb{N} \) such that \( f(x') = x' + \delta \).

**Lemma 26.** The length of any retour \((r,x',b) \to_0^\delta (r',f(x'),b)\) (and similarly for one on the vertical belt) is bounded. Thus we can consider a retour as a bounded segment.
Lemma 27. Given a traversal \((r, x', b) \rightarrow_0^* (r', b, f(x'))\) (and similarly one that goes from the vertical belt to the horizontal belt) we can suppose that it uses just one elementary cycle repeatedly, and has \(f(x) = \rho x' + h(x \mod d)\), for some suitable \(d\).

From Lemma 24, we can see that \(\text{Seq}\) consists of boundedly many tours of three types, i.e. traversals, retours and bounded segments in between, where all bounds do not depend on \(x\). This shows that we have only finitely many patterns for \(w(x)\), depending on how to arrange the three types of tours a bounded number of times, and the values of non-resettable places, i.e. \(r\) in between each tour (which is also bounded) - along with constraints on the initial and final \(r\).

From Lemmas 25 - 27, we have that there are regular path schemes associated with each types of tours. For each of the (finitely many) patterns for \(w(x)\), we just compose the path schemes corresponding to the tours of each type, to get a path scheme corresponding to it. Thus for each \(x\) we have only a finite number (not depending on \(x\)) of path schemes (all regular) we can choose from. For large \(x\), the path schemes with the largest value of \(\rho\) are surely better than the others since \(h\) is a bounded function.

Now if we have two path schemes with this same maximal \(\rho\); \(f_1(x) = \rho x + h_1(x \mod d_1)\) and \(f_2(x) = \rho x + h_2(x \mod d_2)\), we consider \(f(x) = \rho x + h(x \mod d)\) where \(d = d_1d_2\) and \(h(x \mod d) = \max( h_1(x \mod d_1), h_2(x \mod d_2) )\).

Thus in this way, we will finally have \(f(x) = \rho x + h(x \mod d)\) where \(d\) will be the product of \(d'\)'s of all the path schemes with the maximal \(\rho\).

Also since in bounded-segments, we are using the same paths for each \(x\) and in retours, we are using a bounded prefix and suffix along with the same elementary cycle many times (the number of times being an affine function of \(x\)), we have that \(w(x)\) also follows the pattern required by a regular path scheme.

Thus we have shown the existence of a regular path scheme with order 0. Now, for any general number of resets \(n > 0\), we have only finitely many possibilities on the resets, i.e. the number of resets (which is less than or equal to \(n\)), the choice whether each reset is at place \(p_1\) or \(p_2\) and the values of non-resettable places after each reset. In between two resets we have a unique regular path scheme of order zero. For example, between two consecutive resets of \(p_2\) (with values at non-resettable places as \(r_1\) and \(r_2\)), we consider the regular path scheme associated with the pair \(r_1, r_2, 0\). The use of don’t care is in the fact that the place with the don’t care is reset just after this and so we don’t care about its value. Also note that we can compose these path schemes to get finitely many possible path schemes of order \(n\) (the reset allows composition since we just add the reset transition to \(w(x)\); and it has no effect on composing \(f(x)\) for each path scheme since reset always occurs on the don’t care place). Now, from all these possible (regular) path schemes we proceed in the same manner taking the path schemes with the maximal \(\rho\) and product of the corresponding \(d'\)’s.

Hence we conclude the proof of Lemma 15.
Proof of Lemma 24. Let \( k = \lfloor fcs \rfloor \). We suppose that \( x_m > b \) and \( y_m > b \) (where \( b = k! \)). Consider the region \( T = \{(x, y) : x > b/2 \text{ and } y > b/2\} \). Since in one transition we can increase (or decrease) the value at any counter by at most 1 (we have no resets now), the end part of \( w(x) \) must be completely in the region \( T \). Also see that since \( b >> k \), the length of this part is greater than \( k \), and by pigeon-hole principle we can find two points with the same \( r \). Hence we have found a cycle. We can suppose this to be elementary (or we consider an elementary cycle that is a part of this cycle).

(1) If for the cycle, we have \( \Delta x > 0 \) and \( \Delta y < 0 \) : If the path scheme is related to the pair \( \langle - , p_2 \rangle \) we can add one more of this cycle and if it is related to \( \langle - , p_1 \rangle \) we remove this cycle. We note that \( |\Delta x| \leq k \) and \( |\Delta y| \leq k \) and we are in the region \( T \), thus adding or removing such a cycle would keep all subsequent \( x \) and \( y \) also positive. In doing this we are able to increase \( f(x) \) which is not possible.

(2) We can similarly argue for the case \( \Delta x < 0 \) and \( \Delta y > 0 \).

Proof of Lemma 25. We consider a bounded segment \( (r, x', b_1) \rightarrow^* (r', f(x'), b_2) \). Let the length of the segment be bounded by \( B \in \mathbb{N} \). Let \( x_0' = b + B \). Consider arbitrary \( x_1' \geq x_0' \) and \( x_2' \geq x_0' \). Then we have \( (r, x', b_1) \xrightarrow{w(x_1')-0} (r', f(x_1'), b_2) \) and \( (r, x', b_1) \xrightarrow{w(x_2')-0} (r', f(x_2'), b_2) \). We have:

(1) \( (r, x', b_1) \xrightarrow{w(x_1')-0} (r', f(x_1') + x_1' - x_2', b_2) \) - note that we are at large enough \( x \) to be assured that it is possible to fire all transitions. Using the maximal property we have \( f(x_1') \geq f(x_2') + x_1' - x_2' \).

(2) Similarly we will have \( f(x_2') \geq f(x_1') + x_2' - x_1' \).

Thus we will have \( f(x_1') - x_1' = f(x_2') - x_2' \) for arbitrary \( x_1' \geq x_0' \) and \( x_2' \geq x_0' \). Hence we have \( f(x') = x' + \delta \) for some \( \delta \in \mathbb{N} \) and \( x' \geq x_0' \).

Proof of Lemma 26. We consider an arbitrary retour \( (r, x, b) \rightarrow^* (r, f(x), b) \) (we use \( x \) instead of \( x' \) for simplified notation but don’t confuse this with the initial \( x \) we took).

For the retour we have \( \Delta x_{\text{total}} = f(x) - x \) and \( \Delta y_{\text{total}} = 0 \). We fix this given \( x \). We call \( w(x) \) as the sequence of transitions that maximizes \( f(x) \) and assume that \( w(x) \) is the shortest among all those possible.

Let \( k = \lfloor fcs \rfloor \). Then we can find at least one elementary cycle in \( k + 1 \) consecutive markings.

A key point to note is that we can add and remove elementary cycles since we are outside the \( b \) band where \( b \) is very large, and each elementary cycle produces only a small change in \( x \) and \( y \).

We follow these steps:

(1) We find many such elementary cycles and break \( w(x) \) into disjoint elementary cycles and transitions in between them (the length of neither exceeding \( k \)). Let the changes in \( x \) and \( y \) by the cycles be denoted by the sequence \((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_t, \beta_t)\) where \( t \) is the total number of elementary cycles. Note that we also have that \( \alpha_i / \beta_i < 0 \) for all \( 1 \leq i \leq t \) (see Definition 19).

(2) If we have two cycles such that \( \beta_i = \beta_j \) but \( \alpha_i \neq \alpha_j \) (say \( \alpha_i > \alpha_j \)). Then we can remove the \( i^{th} \) cycle and add one more of the \( i^{th} \) cycle. This keeps \( \Delta y_{\text{total}} \) the same (= 0) but increases \( f(x) \). Hence this is not possible. Now we can say \( \beta_i = \beta_j \implies \alpha_i = \alpha_j \). Thus we consider the sequence of changes \((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\), \( n \leq 2k + 1 \), \(-k \leq \alpha_i \leq k \) and \(-k \leq \beta_i \leq k \) for all \( i \), where each change may occur many times (maybe at different places in the path \( w(x) \)). Also we only consider those changes which can occur unboundedly many times, as if all changes occurred boundedly many times then \( w(x) \) would be bounded.

(3) If we have two changes such that \( \beta_i > 0 \) and \( \beta_j > 0 \) and both occur unboundedly many times.

(a) If \( \alpha_i / \beta_i > \alpha_j / \beta_j \) : We add \( \beta_i \) cycles of type \( i \) and remove \( \beta_i \) cycles of type \( j \) (can do so since the number is unbounded now). The total change in \( y \) is zero (also note that coordinates always remove positive always since we are removing at most \( k \) cycles
each of length at most $k$ and $b >> k^2$). The total change in $x$ is $\alpha_i \beta_j - \alpha_j \beta_i$ which is positive. Thus we are able to increase $f(x)$.

(b) We can similarly deal with the case $\alpha_i / \beta_i < \alpha_j / \beta_j$.

(c) If $\alpha_i / \beta_i = \alpha_j / \beta_j$. We again do the same adding and removing of cycles as in the first step. We keep $f(x)$ the same but we are able to bound the number of cycles of type $j$.

Thus only only change with $\beta > 0$ can occur unboundedly many times. Similarly we can argue for $\beta < 0$.

(4) If we have two changes such that $\beta_i > 0$ ($\alpha_i < 0$) and $\beta_j < 0$ ($\alpha_j > 0$) and both occur unboundedly many times.

(a) If $\alpha_i \beta_i > \alpha_j \beta_j$. We add $|\beta_j|$ cycles of type $i$ and add $\beta_i$ cycles of type $j$. The total change in $y$ is zero and total change in $x$ is $\alpha_i |\beta_j| + \alpha_j \beta_i$ which is positive.

(b) If $\alpha_i / \beta_i < \alpha_j / \beta_j$. We remove $|\beta_j|$ cycles of type $i$ and remove $\beta_i$ cycles of type $j$ which makes the total change in $y$ as zero and the total change in $x$ as positive.

(c) If $\alpha_i / \beta_i < \alpha_j / \beta_j$. We do the same as in second step and we are able to shorten $w(x)$ keeping same $f(x)$.

Thus we can have unboundedly many changes of only one sign.

(5) We have only one type of change that occurs unboundedly many times. Let it be $(\alpha, \beta)$. These changes may occur at different $r$'s (the non-resettable marking) in the path $w(x)$. What we do is remove all such cycles from all places and just concentrate all of them at the same $r$ (without worrying about $x$ or $y$ going negative right now). This may generate more elementary cycles and hence changes whose number along the path $w(x)$ may become greater than a bound.

(6) Now we keep on repeating steps (1) - (5) until we finally have only one change that occurs unboundedly many times and the corresponding cycle is concentrated at the same place. This process terminates since we never increase the length of the path $w(x)$ and in each iteration we increase the number of transitions that are a part of the concentrated group of cycles (or we terminate).

Finally what we obtain is only one cycle that occurs unboundedly many times with a bounded prefix and suffix. But note that if this is the case, when the number of times that the cycle that occurs unboundedly increases, we will not be able to obtain $\Delta y_{\text{total}} = 0$.

Thus lengths of returns must be bounded.

**Proof of Lemma 27.** We consider an arbitrary traversal $(r, x, b) \rightarrow^*_\alpha (r, b, f(x))$.

For the traversal we have $\Delta x_{\text{total}} = b - x$ and $\Delta y_{\text{total}} = f(x) - b$. We call $w(x)$ as the sequence of transitions that maximizes $f(x)$ and assume that $w(x)$ is the shortest among all those possible. We follow all the steps in the same way as in the proof for Lemma 26 but interchange $\Delta x$ and $\Delta y$ on the way since here for a given $x$, we want to keep $\Delta x_{\text{total}}$ constant and increase $\Delta y_{\text{total}}$ as much. Finally, we have that only one cycle (say with change $(\alpha, \beta)$) occurs unboundedly many times and has a bounded prefix and suffix. We must have $\alpha < 0$ and $\beta > 0$ for $\Delta x_{\text{total}} = b - x < 0$ (note that this also grows with $x$ unlike in the case of returns). Let $\rho = \beta / |\alpha|$. Since the suffix and prefix are both bounded we must have $f(x) = \rho x + h$ where both $\rho$ and $h$ may depend on $x$. We see that $\rho$ can have only finitely many values since $0 \leq |\alpha| \leq k$ and $0 \leq \beta \leq k$. Consider that $\rho' = \rho' / |\alpha'|$ is the maximum among all $x$ and occurs for some $x = x'$ (i.e., $f(x') = \rho' x' + h'$). Now we can add any number (say $\eta$) of these cycles and start from $x'' = x' + \eta |\alpha'|$ making this a valid path (though not necessarily maximal). We must have $f(x'') \geq \rho' x'' + h'$ by the maximal property.

Take any $x$ which is large. We consider $x''$ such that $0 \leq x - x'' < |\alpha'|$ such that $(x'' \mod |\alpha'|) = (x' \mod |\alpha'|)$. Using Lemma 9 we have $f(x) \geq f(x'')$. Noting that $f(x) = \rho x + h$ for some $\rho$ and $h$, we get $\rho x + h \geq \rho' x'' + h' > \rho' (x - |\alpha'|) + h'$. This implies $(\rho - \rho') x + (h - h' + \rho' |\alpha'|) \geq 0$ for all $x$. The term in the second bracket is bounded (bound not depending on $x$) and so we must have $\rho = \rho'$.

Finally, we have that $f(x) = \rho x + h(x)$ where $h$ is a finite function of $x$ and $\rho = \beta / |\alpha|$ is a constant.
At any $x$ we can add one of the cycle with rate $\rho$ and make it a valid path starting from $x + |\alpha|$, or remove one of the cycle with rate $\rho$ and make it a valid path starting from $x - |\alpha|$. Using the maximal property we have: $f(x + |\alpha|) \geq f(x) + \rho|\alpha|$ and $f(x) \geq f(x + |\alpha|) - \rho|\alpha|$. This gives us $f(x + |\alpha|) = f(x) + \rho|\alpha|$ and substituting values we get $h(x + |\alpha|) = h(x)$. Thus $h$ is a function of $(x \mod |\alpha|)$. Letting $d = |\alpha|$ gives us $f(x) = \rho x + h(x \mod d)$ which completes the proof.

We can also see that $w(x)$ uses the one elementary cycle repeatedly (the number of times being an affine function of $x$) satisfying the property required by a regular path scheme.

### 3. Place Boundedness

This section aims to give a brief idea (informal) regarding the proof in [1] that shows the decidability of place-boundedness in 2-Reset petri Nets.

Section 2 defines acceleration strategies in terms of computable and monotonic relations. It then defines the strategy $\text{ITER}$ and shows how $\text{ITER}^\infty$ works as an acceleration strategy for WSTS. Then a condition is proven in Lemma 2.12 which guarantees the termination of building a Karp-Miller tree using this strategy.

In Section 4, a regular loop is defined and its existence is proven using the regular path scheme we showed in Section 2.3 in this report. We are then able to use this regular loop (along with normal transitions that may lead to a simple loop - similar to a simple witness) as the computable monotonic function required by an acceleration strategy. It is shown that this strategy follows the condition of Lemma 2.12 which makes sure that a Karp-Miller tree can be built, i.e. the algorithm terminates on nets with 2 Reset arcs. Hence the cover is computable and place-boundedness is shown to be decidable.

### 4. Conclusion

In this report, we have completed the proof in Section 4 of [3] regarding the decidability of boundedness for 2-Reset Nets. In Section 3, we gave an idea about the interesting proof in Section 4 in [1] regarding the decidability of place-boundedness for the same.

Finally, we leave the following open problems for future work:

- Given a net with reset arcs, can we decide whether there is a simple witness and/or whether there is a regular witness?
- What are the bounds for complexity of both these problems, i.e. boundedness and place-boundedness.

### References

