# Game Theoretic Verification of Timed Systems

**First Annual Progress Seminar - Literature Survey**

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# Overview

- *Introduction*
- *Basics of games*
- *Untimed Parity Games*
- *Timed Games*
- *Problems to ponder over*

# Introduction

- Open reactive systems - modules with different objectives
- Games - natural way to model such systems
- Verification of safety, reachability - directly translated to corresponding games
- Possible to verify for other objectives - Büchi, Parity, Rabin, Streett
- Existence of equilibria
- Controller synthesis

# Graph Notations

1. *Graph* $G = (V, E)$ where
   $V$ = finite set of vertices
   $E \subseteq V \times V$ = set of edges.
2. Set of successors of a vertex $vE = \{u \mid (v, u) \in E\}$
3. *Path* $\pi = v_0 v_1 v_2 \cdots v_n \cdots$
   such that $\forall i,\ v_i \in V$ and $\forall i, (v_i, v_{i+1}) \in E$
4. $Occ(\pi) = \{v \mid \exists i, \pi_i = v\}$ - the set of vertices visited.
5. $Inf(\pi) = \{v \mid \forall i, \exists j > i, \pi_j = v\}$ - the set of vertices visited infinitely often.

# Graph Game definition

A graph game is $\mathcal{G} = (V, E, P, \{V_i\}_{i \in P}, \{O_i\}_{i \in P})$ where

1. $V$ = finite set of vertices,
2. $E \subseteq V \times V$ = finite set of edges,
3. $P$ = finite set of players,
4. $V_i \subseteq V$ = set of vertices of player $i$.
   Note that the set $V$ is partitioned among the players.
5. $O_i \subseteq V^\omega$ the winning objective for the player $i$
   = the set of plays which are *favorable* to her.

A *play* $\pi$ of the game is an infinite path in the arena $(V, E)$.

# $\omega$-regular winning conditions

$F_i \subseteq V$ ($\mathcal{F}_i \subseteq \mathcal{P}^V$) the set (family of sets) of states desirable to $i$.

A play $\pi = v_0 v_1 v_2 \ldots$ is in $O_i$ if

- *Reachability* - iff $\exists i, v_i \in F_i$
- *Safety* - iff $\forall i, v_i \in F_i$
- *Büchi* - iff $Inf(\pi) \cap F_i \neq \phi$
- *Muller* - iff $Inf(\pi) \in \mathcal{F}_i$
- *Parity* - $\chi : V \rightarrow C$ where $C$ is a set of positive integers. A play is in $O_i$ iff the maximum priority in $Inf(\pi)$ is even.

# Strategy and Winning sets

- A *strategy* for player $i$ is $\sigma_i : V^* V_i \rightarrow V$.
  Helps $i$ win the plays.
- $outcome(v, \sigma_0, \sigma_1, \cdots)$ is the play starting at $v$ and player $i$ adopts strategy $\sigma_i$.
- $\sigma_i$ is *winning* in a vertex $v \in V$
  if any play starting at $v$ agreeing with $\sigma_i$ is in $O_i$
  irrespective of the actions of the other players.
- A set $W_i \subseteq V$ is the *Winning set* for $i$ if she has a winning strategy in all the vertices of $W_i$.
- Memoryless, finite memory or infinite memory strategy.

# Important notions of a game



A game is called

- *zero-sum* if $\forall i, j,\ O_i \cap O_j = \emptyset$.
- *determined* if for a given vertex $v \in V$, exactly one of the players wins the play starting at $v$
  i.e; $\bigcup_i W_i = V$.
- *memoryless determinacy* - if it is determined and all players have memoryless winning strategies.

Henceforth, focus on 2-player, turn based zero-sum games consisting of infinite plays on a finite arena.

# Results for $\omega$-regular winning conditions

- All $\omega-$regular games can be reduced to parity games.
- *Parity games enjoy memoryless determinacy.*
- Until recently, best known complexity to solve parity games - *exponential* in the number of vertices.

# Attractors [2]



$Attr_i(\mathcal{G}, X) =$ *Attractor set of player $i$ for set $X$*
from which $i$ can force a visit to some vertex in $X$

For some $Y \subseteq V$,
$pre_i(Y) = \{v \in V_i | vE \cap Y \neq \phi\} \cup \{v \in V_{1-i} | vE \subseteq Y\}$.

Construction of $Attr_0(\mathcal{G}, X)$ for some $X \subseteq V$.

- $Z_0 = X$
- $Z_i = Z_{i-1} \cup pre_0(Z_{i-1})$
- $Attr_0(\mathcal{G}, X) = Z_i$ for the smallest $i$ such that $Z_i = Z_{i+1}$.

# Rank based memoryless strategy



- *Rank r* -
  $r(s) = 0$ if $s \in X$,
  else $r(s) = j$ such that $s \in Z_j \setminus Z_{j-1}$.
- *Memoryless strategy $\sigma_i$* to reach $X$ from $Attr_i(\mathcal{G}, X)$
  $\sigma_i(s) = s'$ such that rank of $s'$ is lower than that of $s$.
- Memoryless determinacy for reachability, safety, Büchi, Muller and parity games - Based on attractors and rank based strategies

# Results related to attractors

- *Trap $T$ for player $i$* is such that
  $i$ cannot force a visit to $V \setminus T$.
- A set $X \subseteq V$ is called as a *0-paradise* iff
  $X$ is a trap for 1 and
  0 has a memoryless winning strategy in $X$.

## Lemma

1. *The set $V \setminus Attr_i(\mathcal{G}, X)$ is a trap for player $i$.*
2. *The set $Attr_i(\mathcal{G}, X)$ is a trap for player $1 - i$.*
3. *If $U$ is a 0-paradise then so is $Attr_0(\mathcal{G}, U)$*
4. *A union of 0-paradises is also a 0-paradise*

*Note that the same is true for 1-paradises. [[2] lemma 6.5]*

# Subgame

Given a set $X \subseteq V$, *subgame* $\mathcal{G}[X]$ is a game such that

1. the arena $\mathcal{A}_X = (V \cap X, E \cap (X \times X))$ ,
2. every deadend in $\mathcal{G}$ is also a deadend in $\mathcal{G}[X]$,
3. winning conditions consist of plays restricted to vertices in the new arena.

# Results related to subgames

1. *[Transitivity of games]*
   Let $U$ and $U'$ be subsets of $V$ such that
   $\mathcal{G}[U]$ is a subgame of $\mathcal{G}$ and
   $\mathcal{G}[U][U']$ is a subgame of $\mathcal{G}[U]$ then
   $\mathcal{G}[U']$ is a subgame of $\mathcal{G}$. [book [2] lemma 6.2]
2. *[Trap induces a game]*
   For every 0-trap $U$ in $\mathcal{G}$, $\mathcal{G}[U]$ is a subgame.

# Definition of parity game

A *parity game* $\mathcal{G} = (V_0, V_1, E, \chi)$ where

- $V_i$ is the vertex set of player $i$,
- $E$ is the set of edges and
- $\chi : V \longrightarrow C$ is the priority function assigning priorities from $C$ to vertices.

*Solving of a parity game* given a vertex $v \in V$, is to determine whether player 0 has a winning strategy for the play starting in $v$.

**Solving a parity game in $NP \cap Co - NP$.**

# Results for parity games

Let $V = V_0 \cup V_1$, $|E| = m$, $|V| = n$ and $|C| = d$.

1.
   - Parity games - algorithm to find winning sets [2], [7]
   - Complexity - $O(2^n)$ [oldest known algorithm]
   - Using attractors recursively to find winning sets.
2.
   - *Weak parity games* - maximum priority reached is even [8]
   - Deterministic algorithm for memoryless strategies
   - Complexity - Linear time
   - Based on attractors
3.
   - *Generalized parity games* - Boolean expressions of simple parity conditions as winning objectives [9]
   - Memoryless strategy for disjunction and finite memory strategy for conjunction
   - Complexity -$n^{O(\sqrt{n})}.O(k.d).\ (d_1, d_2, \overset{d}{\cdots} d_k)$
   - Based on attractors

# Results for parity games - II

1.
   - Parity games - deterministic algorithm for strategies [4]
   - Best known complexity- $n^{O(\sqrt{n})}$
   - Based on small sized winning attractors - *dominions*.
2.
   - Parity games - deterministic algorithm for strategies [5]
   - Complexity - $O(dm.(\frac{n}{\lfloor d/2 \rfloor})^{\lfloor d/2 \rfloor})$ time and $O(dn)$ space
   - Based on improvement of small progress measures.
3.
   - Parity games - deterministic algorithm for strategies [6]
   - Complexity - $n^{O((n/d)^d)}$ time
   - Discrete strategy improvement - based on vertex profiles.

# Timed Automata [1]

A *timed automaton $\mathcal{A}$ is a tuple* $(L, L_0, \Sigma, X, E, F)$

- $L$ - Finite set of locations,
- $L_0 \subseteq L$ - Set of initial locations,
- $\Sigma$ - Finite set of symbols (called alphabet),
- $X$ - Finite set of real valued clocks,
- $E \subseteq L \times L \times \Sigma \times \mathcal{C}(X) \times \mathcal{U}(X)$ finite set of transitions.
- $F \subseteq L$ - Set of final locations.
- $\mathcal{C}(X)$ - Set of clock constraints
  of the form $\varphi := x \sim c | \varphi_1 \wedge \varphi_2 | \varphi_1 \vee \varphi_2$
  s.t $x \in X, c \in \mathbf{Q}^+, \sim \in \{=, \neq, <, >, \leq, \geq\}$ and
  $\varphi_1, \varphi_2 \in \mathcal{C}(X)$.
- $\mathcal{U}(X)$ - Set of clock resets $\phi \subseteq X$

# Clock valuations

*Clock interpretation* $\nu : X \rightarrow \mathbf{T}$

- $\mathbf{T}^X$ = set of all clock interpretations.
- $\nu(x)$ is the real value of $x$
- $\nu + t$ : for $t \in \mathbf{T}$ if $\forall x \in X, \nu + t(x) = \nu(x) + t$.
- $\nu \models \varphi$: for $\varphi \in C(X)$
  if $\forall x \in X$, if $\varphi$ has $x \sim c$ then $\nu(x) \sim c$.
- $\phi(\nu)$ : for $\phi \subseteq X$ s.t $\phi(\nu)(x) = 0$
  $x \in \phi$ else $\phi(\nu)(x) = \nu(x)$

# State, Run, Word

- *State*:$(l, \nu)$ s.t $l \in L$ and $\nu \in \mathbf{T}^{|X|}$.
  *State space* of $\mathcal{A}$ is $L \times \mathbf{T}^{|X|}$.
- A *run* $\pi$ of $\mathcal{A}$ is
  $(l_0, \nu_0) \xrightarrow{\tau_1} (l_0, \nu_0 + \tau_1) \xrightarrow{(\sigma_1, \varphi_1, \phi_1)} (l_1, \nu'_1) \xrightarrow{\tau_2} (l_1, \nu'_1 + \tau_2) \cdots$
  $(l_{n-1}, \nu'_{n-1}) \xrightarrow{\tau_n} (l_{n-1}, \nu_n) \xrightarrow{(\sigma_n, \varphi_n, \phi_n)} (l_n, \nu'_n) \cdots.$
- *Word of Run*:
  $\langle (\sigma_1, \tau_1), (\sigma_2, \tau_1 + \tau_2) \ldots (\sigma_n, \tau_1 + \ldots + \tau_n) \rangle \cdots$

# Regions

- $\mathcal{R}$ *- finite set of partitions* $(\alpha)$ of $\mathbf{T}^X$.
  $\alpha = \{\nu | \nu \in \mathbf{T}^X\}$ (pos. infinite set)
- Typically $\alpha = (\{I_x\}_{x \in X}, \prec)$ where
  $I_x = \{[0] to [c_m]\} \cup \{(0,1) to (c_m - 1, c_m)\} \cup (c_m, \infty)$ :
  $c_m =$ max constant in $\mathcal{A}$
  $\prec$ is pre-order on fractional values of clocks.
- $Succ(\alpha)$: $\alpha' \in \mathcal{R}$ if $\exists \nu \in \mathcal{R}, \exists t \in \mathbf{T}$ s.t $\nu + t \in \alpha'$
- $\alpha \models \varphi : \forall \nu \in \alpha, \nu \models \varphi$.
- $\phi(\alpha) = \{\alpha' | \exists \nu \in \alpha, \alpha' \cap \phi(\nu) \neq \emptyset\}$.

# 3 Important Conditions

$\mathcal{R}$ should satisfy all the following:

1. *set of regions* [Time elapse consistency] - two equivalent valuations remain equivalent with time elapse
2. $\mathcal{C}(X)$ *compatible* - all valuations in a region satisfy a constraint or all of them satisfy its negation.
3. $\mathcal{U}(X)$ *compatible* - a reset maps an entire region to another.

# Region Automaton

For $\mathcal{A} = (L, L_0, \Sigma, X, E, F)$
*Region Automaton* $= \mathcal{R}(\mathcal{A}) = (Q, Q_0, \Sigma, E', F')$ where

1. $\mathcal{R}$ : satisfies 3 conditions
2. $Q = L \times \mathcal{R}$ - set of locations
3. $Q_0 \subseteq Q$ - set of initial locations
4. $F' \subseteq Q$ - set of final locations
   where $(l, \alpha) \in F'$ s.t $l \in F \wedge \alpha \in \mathcal{R}$.
5. $E' \subseteq (Q \times \Sigma \times Q)$ - set of edges s.t
   $(l, \alpha) \xrightarrow{a} (l', \alpha') \in E'$ if
   $\exists \alpha'' \in \mathcal{R}$ and $(l, l', a, \varphi, \phi) \in E$ s.t
   - $\alpha'' \in Succ(\alpha)$
   - $\alpha'' \models \varphi$
   - $\alpha' \in \phi(\alpha'')$

$L(\mathcal{R}(\mathcal{A})) = Untime(L(\mathcal{A}))$

# Timed games over timed automaton

- Arena - timed automaton
- Concurrent - edge set $E$ partitioned
  Both players propose $(t_i, a_i)$, the one with smaller delay is chosen.
- Turn based - location set $L$ partitioned
- Winning objective $\Phi$ - set of winning timed plays for 0.
- Notions of strategies, winning sets remain same

## Theorem

*The timed automaton games (and hence, the timed game structures) are neither weakly, not strongly determined for the class of reachability goals* [16].

# Time divergent winning objective - concurrent

$WC_0(\Phi) = (\Phi \cap td) \cup (Blameless_0 \setminus td)$ where

- $\Phi$ - given winning objective - set of winning plays for 0
- $td$ - time divergent plays in the game
- $Blameless_0$ - plays in which 0's moves are chosen only finitely many times

# Results for timed games over timed automata

1.
   - Concurrent parity - compute winning sets [16]
   - memoryless strategy for safety and
     no memoryless strategy for reachability
     (infinite memory for precise clock)
   - Complexity - $O(|L|.|X|!.2^{|X|}.(2c_m + 1)^{|X|}.|H|.|H_*|^{|H|+1})$ time
   - Symbolic algorithms working on timed automaton and objective as parity automaton with $H$ states.
2.
   - Concurrent Parity - winning sets and strategy [17]
   - memoryless strategy for safety and
     finite memory randomized strategy for reachability
   - Complexity - EXPTIME complete
   - Parity game is solved as reachability and safety objectives applied alternately
     Strategy selects randomized time delays.

# Results for timed games over timed automata- II

1.
   - Concurrent Parity - Winning sets and strategy [12]
   - Untime the game
   - Complexity - resulting number of states = $|Q|.(1 + (c_m + 1.(|X| + 2).2.(E_1 + 1)))$
   - Reduction to untimed turn-based parity game - based on region automata
2.
   - Concurrent timed game - calculating minimum time to reach target states [14]
   - Receptive strategy
   - Complexity - $O((|L|.(c_m + 1)^{|C|}|C + 1|!2|C|)^2|C|)$ time - EXPTIME complete
   - Based on Fix point $\mu$ expression on extended clock regions

Infinite memory needed in concurrent reachability games.

# Results for timed games on timed automata - III

1.
   - Turn based timed game - calculating time to reach target states [13]
   - Winning strategy and a two functions which determine the time to reach and number of moves to reach
   - Complexity - $O(|\mathcal{R}|^3)$ - EXPTIME complete for 2 clocks
   - Based on Strategy improvement for regionally constant strategies
2.
   - Turn based timed game - players trying to minimise or maximise average time/edge [15]
   - Reduction to average-price untimed game
   - Complexity - EXPTIME complete for 2 clocks
   - Average-price untimed game based on regions consistent with values (average time)

Finite memory strategies can be built for turn based games.

# Parity games problems - I

1. **Weak Parity Games [8]** - Complexity for a bounded number of visits
2. **Parity games** - Will the complexity reduce if we considered strategies with finite or infinite memory?
3. **Restricted Parity games** - complexity - with some of the parameters (number of priorities, out edges, in edges, no. of vertices, no. of visits, etc) restricted.
4. **Multi player parity games** - definition of the game has to be proposed. Questions similar to parity can be asked here too.
5. **Infinite arena games** - can the currently known techniques (progress measures [5], strategy improvement [6]) be extended to infinite arena games?

# Parity games problems- II

1. **Nash equilibrium, Nash determinacy** - for untimed games with the payoff being simple metrics
2. **Sharing of vertices i.e;** $V_0 \cap V_1 \neq \emptyset$ - At some vertices it is turn based and at others it is concurrent. pick some criterion to decide whose move is picked. This should be interesting cause it is too strict to declare that only one player has a say.
3. **Deterministic sub-exponential algorithm for solving parity games [4]** - based on dominions of size $l = \lceil \sqrt{2n} \rceil$. tweak $l$ for better complexity?
4. **Probabilistic turn based parity games** - each player has a probability distribution over the outgoing edges from her vertex.

# Timed games problems - I

1. **Concurrent probabilistic times games** - enriching games in [11] with time.
2. **Complexity of timed Büchi games** - turn based games with Büchi objective.
3. **Problems of parity with time** - add timing dimension to parity problems in earlier section.
4. **Coalitional timed games** - non-zero sum games.
5. **Hybrid arena for timed games** - some of the locations as concurrent and some as turn based.

# Timed games problems - II

1. **Arena defined on the states as opposed to locations** - ex : for a location, the invariants under which player *i* can choose the outgoing edge.
2. **Timing constraints in the goals** - so far most of the research has been aimed at location goals. [18]
3. Do the results vary with the number of clocks - based on region equivalence
4. **Equilirbia** - There are some papers dealing with time as a metric, can we formulate notions of equilibria for these games?

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