Model Checking Weighted Timed Automata

Seminar Report

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Abstract

Weighted timed automata (WTA), introduced in [3], [11] are an extension of [1] timed automata, a widely accepted formalism for the modelling and verification of real time systems. Weighted timed automata extend timed automata by allowing costs on the locations and edges. There has been a lot of interest [16], [17], [12], [15] in studying the modelchecking problem of weighted timed automata. The properties of interest are written using logic weighted CTL (WCTL), an extension of CTL with costs. It has been shown [15] that the problem of modelchecking for WTAs with a single clock and cost using WCTL without external cost variables is decidable, while 3 clocks render the problem undecidable [12]. The question of 2 clocks is open.

In this report, we give a short survey of the model checking results for WTAs studied so far in literature. Further, we introduce a subclass of weighted timed automata called weighted integer reset timed automata (WIRTA) and study the model checking problem. We give a clock reduction technique for WIRTA. Given a WIRTA $\mathcal{A}$ with $n \geq 1$ clocks, we show that a single clock WIRTA $\mathcal{A}'$ preserving the paths and costs of $\mathcal{A}$ can be obtained. This gives us the decidability of modelchecking WIRTA with $n \geq 1$ clocks and $m \geq 1$ costs using WCTL with no external cost variables. We then show that for a restricted version of WCTL with external cost variables, the model checking problem is undecidable for WIRTA with 3 stopwatch costs and 1 clock.

1 Introduction

Timed automata [1] are a well-established model for real-time systems. One of the most important properties of timed automata is that reachability is decidable. This has paved the way for using timed automata in verification and many tools viz., UPPAAL [10], KRONOS [19] and HyTech [5] were built using this. Since their inception, several variants of timed automata have been considered: [13], wherein new operations on clock updates were considered, [8] where new kinds of guards were introduced, [20] where a set of clocks could be freezed, [9], where silent transitions were studied, and [23], where new kinds of updates along with additive and diagonal constraints were studied. However, timed automata are not considered adequate enough for representing properties of systems, since they are not closed under complementation, and their inclusion problem is not decidable [1]. Hence, the quest for finding a more attractive class of timed automata has been going on. Some of the interesting classes here are : (i) [2] event clock automata, (ii) [22] open and closed automata, (iii) [4] perturbed timed automata, (iv) [18] timed automata with periodic constraints, (v) [26] one clock timed automata, and (vi) timed automata with integer resets introduced in [25] and subsequently investigated in [27].

An extension of timed automata useful in applications like scheduling problems and controller synthesis is weighted timed automata (WTA) introduced in [3], [11]. In this, cost variables are attached to locations and edges. The costs are used only as observers, they cannot be compared or evaluated in the automata. The behaviour of the automata is not influenced by these costs. This has led to several decidability results for optimization problems like minimum cost reachability, cost optimal schedules and so on. A generalization of these questions is the model checking problem of WTAs using cost extended versions of linear as well as branching time logics. The logics WCTL and WMTL are considered as cost extensions of logics CTL and LTL (like TCTL, MTL which are timed extensions of CTL, LTL) and are interpreted over WTAs. While it is known [6] that TCTL model checking of timed automata is decidable, it has already been shown that WCTL model checking of WTAs is not decidable [17], [12]. [17] shows that with 3 stopwatch costs and 1 clock, model checking WCTL with external cost variables is undecidable, while in the case of 2 clocks and 1 stopwatch cost or 2 stopwatch costs and 1 clock, it is possible to obtain an infinite bisimulation. [12] has shown that if there are 3 clocks and 1 stopwatch cost, model checking WCTL with no external cost variables is undecidable. The case of model checking WTAs with 2clocks and 1 stopwatch cost has remained an open problem.

In this report, we adress this issue by studying the model checking question on a sub class of WTAs called weighted integer reset timed automata (WIRTA). These are extensions of IRTAs introduced in [25]. It has been shown [25], [27] (albeit with different complexities, non-primitive recursive [25] versus doubly exponential [27]) that if $\mathcal{A}$ is a timed automaton and if $\mathcal{B}$ is an IRTA, then the question $L(\mathcal{A}) \subseteq L(\mathcal{B})$? is decidable. We show that it is possible to reduce the number of clocks in a given WIRTA to one. This, along with the decidability result [15] of model checking WCTL with no external cost variables on one clock WTAs gives us the result that model checking WCTL with no external cost variables is decidable for WIRTAs with n clocks and m costs, $n, m \geq 1$. We then investigate model checking WCTL with external cost variables on WIRTAs. Here, we obtain an undecidability result with 3 stopwatch costs and 1 clock, on a restricted version of the logic (called $WCTL_r$ in [17]).

The organisation of the report is as follows as follows: Section 2 gives the prerequisites, introduces WTAs and gives the syntax, semantics, expressiveness and model checking results so far for logic WCTL, section 3 introduces WIRTA, section 4 gives the clock reduction results and section 5 talks about the undecidability result, and we conclude in section 6.

2 Prerequisites

For any set S, S^* denotes the set of all strings over S. We consider as time domain $\mathbb{T}$ the set $\mathbb{Q}_+$ or $\mathbb{R}_+$ of non-negative rationals or reals, and Σ a set of actions. A time sequence over $\mathbb{T}$ is a non-decreasing sequence $\tau = (t_i)_{i\geq 1}$; for simplicity t_0 is taken to be zero always. Let X be a set of clocks. A clock valuation over X is a mapping $\nu : X \rightarrow \mathbb{R}_+$. We denote by $\mathbb{R}_+^X$ (or $\mathbb{T}^X$) the set of clock valuations over X. If $\nu \in \mathbb{T}^X$ and $\tau \in \mathbb{T}$, then $\nu + \tau$ is the clock valuation defined by $(\nu + \tau)(x) = \nu(x) + \tau$, for $x \in X$. A guard or constraint over X is a conjunction of expressions of the form $x \sim c$ where $x \in X$, $c \in \mathbb{N}$ and $\sim \in \{<, \leq, >, \geq, =\}$. We denote by $\mathcal{G}(X)$ the set of guards over X. The satisfaction relation for guards over clock valuations is denoted as $\nu \models g$ whenever valuation ν satisfies guard g in the usual way.

Clock constraints allow us to test the values of clocks. To change the value of a clock x we use clock resets. $U_0(X)$ denotes the set of resets $\phi \in U_0(X)$ defined as $\phi \subseteq X$.

Let ν be a valuation and let up_z be a simple reset over clock z. A valuation ν' is in $up_z(\nu)$ if $\nu'(y) = \nu(y), y \neq z$ and $\nu'(z) = 0$. For $\phi \subseteq X$, we use the notation $\nu' = \nu[\phi := 0]$ to denote $\nu'(z) = \nu(z)$ for all $z \in X \backslash \phi$ and $\nu'(y) = 0$ for all $y \in \phi$.

2.1 Timed Automata

A *timed automaton* [1] is a tuple $\mathcal{A} = (L, L_0, \Sigma, X, E, F)$ where L is a set of locations; $L_0 \subseteq L$ is a set of initial locations; Σ is a set of symbols; X is a set of clocks; $E \subseteq L \times L \times \Sigma \times \mathcal{G}(X) \times U_0(X)$ is the set of transitions and $F \subseteq L$ is a set of final locations. $\mathcal{G}(X)$ and $U_0(X)$ are the set of clock constraints and clock resets as described above. An edge $e = (l, l', a, \varphi, \phi)$ represents a transition from l to l' on symbol a, with the valuation $\nu \in \mathbb{T}^X$ satisfying the guard φ, and then ϕ gives the resets of certain clocks.

A path is a finite (infinite) sequence of consecutive transitions. The path is said to be accepting if it starts in an initial location ($l_0 \in L_0$) and ends in a final location (or repeats a final location infinitely often). A run through a path from a valuation ν'_0 (with $\nu'_0(x) = 0$ for all x) is a sequence $(l_0, \nu'_0) \xrightarrow{t_1} (l_0, \nu_1) \xrightarrow{(\sigma_1, \varphi_1, \phi_1)} (l_1, \nu'_1) \xrightarrow{t_2} (l_1, \nu_2) \xrightarrow{(\sigma_2, \varphi_2, \phi_2)} (l_2, \nu'_2) \cdots (l_n, \nu'_n)$. Note that $\nu_i = \nu'_{i-1} + (t_i - t_{i-1}), \nu_i \models \varphi_i$, and that $\nu'_i = \nu_i[\phi_i := 0], i \geq 1$. A timed word ρ is accepted by $\mathcal{A}$ iff there exists an accepting run (through an accepting path) over $\mathcal{A}$, the word corresponding to which is ρ. The timed language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is defined as the set of all timed words accepted by $\mathcal{A}$.

2.2 Region Automata

Given a set X of clocks, let $\mathcal{R}$ be a partitioning of $\mathbb{T}^X$. Each partition contains a set (possibly infinite) of clock valuations. Given $\alpha \in \mathcal{R}$, the successors of α

represented by $Succ(\alpha)$ are defined as

$$\alpha' \in Succ(\alpha) \text{ if } \exists \nu \in \alpha, \exists t \in \mathbb{T} \text{ such that } \nu + t \in \alpha'$$

The partition $\mathcal{R}$ is said to be a *set of regions* iff

$$\alpha' \in Succ(\alpha) \iff \forall \nu \in \alpha, \exists t \in \mathbb{T} \text{ such that } \nu + t \in \alpha'.$$

A set of regions is consistent with time elapse if two valuations which are equivalent (within the same partition) stay equivalent with time elapse. A region $\alpha \in \mathcal{R}$ is said to satisfy a clock constraint $\varphi \in \mathcal{G}(X)$ denoted as $\alpha \models \varphi$, if $\forall \nu \in \alpha, \nu \models \varphi$. A clock reset $\phi \in U_0(X)$ maps a region α to a regions $\alpha[\phi := 0] = \alpha'$ such that $\alpha' \cap \nu[\phi := 0] \neq \emptyset$ for some $\nu \in \alpha\}$.

A set of regions $\mathcal{R}$ is said to be *compatible* with a set of clock constraints $\mathcal{G}(X)$ iff $\forall\ \varphi \in \mathcal{G}(X)$ and $\forall \alpha \in \mathcal{R}$ either $\alpha \models \varphi$ or $\alpha \models \neg\varphi$. A set of regions $\mathcal{R}$ is said to be *compatible* with a set of clock resets $U_0(X)$ iff $\alpha' = \alpha[\phi := 0] \Rightarrow \forall \nu \in \alpha, \exists \nu' \in \alpha'$ such that $\nu' \in \nu[\phi := 0]$.

Given a timed automaton $\mathcal{A}$, and a set of regions $\mathcal{R}$ compatible with $\mathcal{G}(X)$ and $U_0(X)$, the *region automaton* $\mathcal{R}(\mathcal{A}) = (Q, Q_0, \Sigma, E', F')$ is defined as follows: $Q = L \times \mathcal{R}$ the set of locations; $Q_0 = L_0 \times \mathcal{R} \subseteq Q$ the set of initial locations; $F' = F \times \mathcal{R} \subseteq Q$ the set of final locations; $E' \subseteq (Q \times \Sigma \times Q)$ is the set of edges. $(l, \alpha) \xrightarrow{a} (l', \alpha')$ if $\exists \alpha'' \in \mathcal{R}$ and a transition $(l, l', a, \varphi, \phi) \in E$ such that (a) $\alpha'' \in Succ(\alpha)$, (b) $\alpha'' \models \varphi$ and (c) $\alpha' = \alpha''[\phi := 0]$.

The region automaton is an abstraction of the timed automaton accepting Untime($L(\mathcal{A})$) [1].

Theorem 1. *Let $\mathcal{A}$ be a timed automaton. Then the problem of checking emptiness of $L(\mathcal{A})$ is decidable.*

2.3 Weighted Timed Automata

Let Σ be a set of atomic propositions. We recall the definition of WTAs [16]. A *weighted timed automaton* is a tuple $\mathcal{A} = (L, L_0, X, Z, E, \theta, \eta, C)$ where L is a set of locations, $L_0 \subseteq L$ is a set of initial locations, X is a set of clocks, Z is a set of costs where $|Z| = m$, $E \subseteq L \times \mathcal{G}(X) \times U_0(X) \times L$ is the set of transitions. A transition $e = (l, \varphi, \phi, l') \in E$ is a transition from l to l' with valuation $\nu \in \mathbb{T}^X$ satisfying the guard φ, and ϕ gives the set of clocks to be reset. $\theta : L \to 2^{\Sigma}$ where θ is the labelling function which associates with each location a subset of Σ. $\eta : L \to \mathcal{G}(X)$ defines the invariants of each location. $C : L \cup E \to \mathbb{N}^m$ is the cost function which gives the rate of growth of each cost. Note that the costs are called *stopwatches* if $C : L \cup E \to \{0, 1\}^m$. From the nature of the costs and stopwatches, it is clear that stopwatches are restricted costs. Hence, WTA with stopwatches form a subclass of WTA with costs.

The semantics of a WTA $\mathcal{A} = (L, L_0 X, Z, E, \theta, \eta, C)$ is given by a labelled timed transition system $\mathcal{T}_\mathcal{A} = (S, \rightarrow)$ where $S = L \times \mathbb{T}^X \times \mathbb{T}^Z$ and $\rightarrow$ is composed of transitions

- Time elapse t in l: $(l, \nu, \mu) \xrightarrow{t} (l', \nu', \mu')$, $t \in \mathbb{T}$. Then $l' = l$, $\nu' = \nu + t$, $\mu' = \mu + C(l) * t$ and for all $0 \leq t' \leq t$, $\nu + t' \models \eta(l)$.
- Location switch: $(l, \nu, \mu) \xrightarrow{(\varphi,\phi)} (l', \nu', \mu')$ if there exists $e = (l, \varphi, \phi, l') \in E$, such that $\nu \models \varphi$, $\nu' = \nu[\phi := 0]$ and $\mu' = \mu + C(e)$. Here, $\nu \models \eta(l)$, $\nu' \models \eta(l')$.

A path is a sequence of consecutive transitions in the transition system $\mathcal{T}_\mathcal{A}$. A path ρ starting at (l_0, ν'_0, μ'_0) is denoted as $\rho = (l_0, \nu'_0, \mu'_0) \xrightarrow{t_1} (l_0, \nu_1, \mu_1) \xrightarrow{(\varphi_1,\phi_1)} (l_1, \nu'_1, \mu'_1) \xrightarrow{t_2} (l_1, \nu_2, \mu_2) \xrightarrow{(\varphi_2,\phi_2)} (l_2, \nu'_2, \mu'_2) \cdots (l_n, \nu'_n, \mu'_n)$. Note that $\nu_i = \nu'_{i-1} + (t_i - t_{i-1}), \nu_i \models \varphi_i$, $\nu'_i = \nu_i[\phi := 0]$ and $\mu_i = \mu'_{i-1} + C(l_{i-1}) * (t_i - t_{i-1})$, $\mu'_i = \mu_i + C(l_{i-1}, \varphi_i, \phi_i, l_i)$. The i^{th} state of the path is denoted as $\rho[i]$ and $\rho[\leq i]$ indicates the prefix of the path till position i. We refer to an element $l \in L$ of a WTA $\mathcal{A}$ as a *loction* while we refer to an element $(l, \nu, \mu) \in S$ of $\mathcal{T}_\mathcal{A}$ as a *state*. The terms transition and edge are used interchangeably.

2.4 Weighted Computational Tree Logic(WCTL)

The logic weighted CTL (WCTL) extends CTL with constraints over costs. Let Z be a set of costs. We now give the syntax of WCTL with external cost variables as in [17]. Let Σ be the set of atomic propositions.

$$\psi ::= true \mid \sigma \mid \pi \mid z.\psi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{E}(\psi \mathbf{U} \psi) \mid \mathbf{A}(\psi \mathbf{U} \psi)$$

where $z \in Z$, $\sigma \in \Sigma$, and π is a cost constraint of the form $z_i \sim c$ or $z_i - z_j \sim c$ for costs $z_i, z_j \in Z$ and $c \in \mathbb{N}$. When the cost constraints π are restricted to be only of the form $z_i \sim c$, then the logic is denoted by $WCTL_r$. The freeze quantifiers $z.$ allows us to reset costs, while the cost constraints $z \sim c$ allows us to test them.

The formulae are evaluated on a WTA. The sets Σ, Z are same for the formula as well as for the WTA.

Given a WTA $\mathcal{A}$, its transition system $\mathcal{T}_\mathcal{A}$ and a WCTL formula ψ, the satisfication relation $\mathcal{A}$,$(l, \nu, \mu) \models \psi$ is defined as follows:

- $\mathcal{A}$,$(l, \nu, \mu) \models \sigma$ iff $\sigma \in \theta(l)$
- $\mathcal{A}$,$(l, \nu, \mu) \models \pi$ iff $\mu \models \pi$
- $\mathcal{A}$,$(l, \nu, \mu) \models \neg\psi$ iff $\mathcal{A}$,$(l, \nu, \mu) \not\models \psi$
- $\mathcal{A}$,$(l, \nu, \mu) \models \psi_1 \vee \psi_2$ iff $\mathcal{A}$,$(l, \nu, \mu) \models \psi_1$ or $\mathcal{A}$,$(l, \nu, \mu) \models \psi_2$.
- $\mathcal{A}$,$(l, \nu, \mu) \models z.\psi$ iff $\mathcal{A}$,$(l, \nu, \mu[z := 0]) \models \psi$ where $\mu[z := 0]$ stands for μ with z reset to zero.
- $\mathcal{A}$,$(l, \nu, \mu) \models \mathbf{E}\psi_1 \mathbf{U} \psi_2$ iff there exists a run ρ in the transition system $\mathcal{T}_\mathcal{A}$ starting at (l, ν, μ), there exists a position i in ρ such that $\rho[i] = (l_i, \nu_i, \mu_i) \models \psi_2$ and forall $j < i, \rho[j] \models \psi_1$.

– $\mathcal{A}$, $(l,\nu,\mu) \models \mathbf{A}\psi_1\mathbf{U}\psi_2$ iff for any run ρ in the transition system $\mathcal{T}_\mathcal{A}$ starting at (l,ν,μ), there exists a position i in ρ such that $\rho[i] = (l_i,\nu_i,\mu_i) \models \psi_2$ and forall $j < i, \rho[j] \models \psi_1$.

Next, we define logic WCTL with no external cost variables [15].

$$\psi ::= true \mid \sigma \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid \mathbf{E}\psi_1\mathbf{U}_{z\sim c}\psi_2 \mid \mathbf{A}\psi_1\mathbf{U}_{z\sim c}\psi_2$$

with $z \in Z$, $c \in \mathbb{N}$, $\sigma \in \Sigma$. Given a WTA $\mathcal{A}$, its transition system $\mathcal{T}_\mathcal{A}$, the semantics of $\mathbf{E}\psi_1\mathbf{U}_{z\sim c}\psi_2$ and $\mathbf{A}\psi_1\mathbf{U}_{p\sim c}\psi_2$ are given below. The semantics of the other formulae are same as described above.

– $\mathcal{A}$,$(l,\nu,\mu) \models \mathbf{E}\psi_1\mathbf{U}_{z\sim c}\psi_2$ iff there exists a run ρ in the transition system $\mathcal{T}_\mathcal{A}$ starting at (l,ν,μ), there exists a position i in ρ, such that $\rho[i] = (l_i,\nu_i,\mu_i) \models \psi_2$ and forall $j < i, \rho[j] \models \psi_1$, with $\mu_i(z) - \mu(z) \sim c$.
– $\mathcal{A}$,$(l,\nu,\mu) \models \mathbf{A}\psi_1\mathbf{U}_{z\sim c}\psi_2$ iff for any run ρ in the transition system $\mathcal{T}_\mathcal{A}$ starting at (l,ν,μ), there exists a position i in ρ, such that $\rho[i] = (l_i,\nu_i,\mu_i) \models \psi_2$ and forall $j < i, \rho[j] \models \psi_1$, with $\mu_i(z) - \mu(z) \sim c$.

Let us denote logic WCTL with no external variables by $WCTL_1$, and WCTL with external cost variables by $WCTL_2$. The restriction of $WCTL_2$ where cost constraints $z_i - z_j \sim c$ are not allowed is denoted by $WCTL_{2r}$. We now compare the expressive power of the logics $WCTL_1, WCTL_2$.

Lemma 1. *$WCTL_{2r}$ is more expressive than $WCTL_1$.*

Proof. We only give a proof sketch. Consider the $WCTL_{2r}$ formula $\psi = z.\mathbf{EF}([a \wedge z \leq 1] \wedge \mathbf{EG}[z \leq 1 \Rightarrow \neg b])$, where $a, b \in \Sigma$. It can proved that there is no $WCTL_1$ formula equivalent to ψ using an argument similar to the one used for showing that TPTL is more expressive than MTL [14]. □

Survey of model checking results

The Table 1 reports the results of model checking $WCTL_1$, $WCTL_2$ and $WCTL_{2r}$ over WTAs with stopwatches. There has been no study of the model checking problem over WTAs with costs which are not stopwatches except the decidability result of [15].

Undecidablity Result : All the undecidability results in the Table 1 are obtained by reducing the halting problem of two counter machine [24] to the model checking problem over WTA.

Decidablity Result - $WCTL_{2r}$ over WTA with one clock and one stopwatch: [16] shows that there exists a finite bisimulation for a WTA with one clock and one stopwatch. This renders the model checking problem over this subclass of WTA decidable.

Logic	Clocks	Stopwatches	Result
$WCTL_1$	1	≥ 1	Decidable [15]
$WCTL_1$	3	1	Undecidable [12]
$WCTL_1$	2	1	Undecidable [21]
$WCTL_2$	1	3	Undecidable [16]
$WCTL_2$	0	3	Undecidable [16] (costs on edges)
$WCTL_{2r}$	1	2	Infinite Bisimulation [16]
$WCTL_{2r}$	2	1	Infinite Bisimulation [16]
$WCTL_{2r}$	1	1	Decidable [16]

Table 1. Model checking over WTAs with stopwatches

Decidablity Result - $WCTL_1$ over WTA with one clock and m costs: Let the problem be model checking $WCTL_1$ formula Φ over WTA $\mathcal{A}$. [15] considers a set of clock regions R over which the truth of Φ is uniform. R is given by $\{[c], (c, c+1) \mid c \in \{0 = a_0, a_1, \cdots a_n, a_{n+1} = \infty\}\}$ such that

- n is finite,
- $a_0 < a_1 < a_2 \cdots a_n < a_{n+1}$,
- $a_1, a_2 \cdots$ are integral multiples of $1/C^{h(\Phi)+1}$,
- C is the l.c.m of all costs labelling a location in $\mathcal{A}$ and
- $h(\Phi)$ is the maximum number of nested constrained modalities in Φ.

[**Note** that in the following argument, the cost valuation in states of transition system of $\mathcal{A}$ is not given any importance as this information is captured over the edges.]

The decidablity result follows from the construction of a graph G whose vertices are (q, r) (q is a location in $\mathcal{A}$ and $r \in R$) and the edge lables are intervals of costs between the vertices. G has additional vertices of the form $(q, \nu(x), r)$ ($\nu(x) \in r$) which indicate the start of a computation. The edges of G are as follows

- edges from $(q, \nu(x), r)$ to (q, r) are labelled with the interval $[0, c)$ if $r = (a_i, a_{i+1})$ else the label is [0,0]. Here c cost accumulated due to $a_{i+1} - \nu(x)$ time elapsed in location q.
- edges from $(q, [a_i])$ to $(q', [a_{i+1}])$: if $q' = q$, then it is labelled with the cost accumulated by 1 t.u elpased in q else it is labelled with the cost accumulated over the path from (q, a_i) to (q', a_{i+1}) in $\mathcal{T}_{\mathcal{A}}$.
- edges incident with $(q, (a_i, a_{i+1}))$ are considered for reset transitions or to finish the computation.

The graph G is such that ρ exists in G iff there is a path ρ' from $(q, \nu(x))$ to $(q', \nu'(x))$ in $\mathcal{T}_\mathcal{A}$ such that $\nu(x) \in r$, $\nu'(x) \in r'$ and the accumulated cost over ρ' is $d \in s$.

The problem $(q, \nu(x)) \models \varPhi$ is determined by the existence of a path ρ from the vertex $(q, \nu(x), r),\ \nu(x) \in r$ to an appropriate vertex (q', r') in G. The sum s of all the cost intervals labelling edges in the path ρ is such that it satisfies the constraints on the modality of $\varPhi$ (other cases not involving modality are trivial).

3 Weighted Integer Reset Timed Automata (WIRTA)

In this section, we introduce a subclass of WTA called weighted integer reset timed automata along the lines of IRTA introduced in [25]. In this subclass of automata, the reset of clocks are restricted to happen only at integer time points. An integer reset timed automaton (IRTA) is a timed automaton in which every edge $e = (l, l', a, \varphi, \lambda)$ is such that λ is nonempty only if φ contains at least one atomic constraint of the form $x = c$, for some clock x, $c \in \mathbb{N}$.

Definition 1. *A Weighted Integer Reset Timed Automaton (WIRTA) is a WTA* $\mathcal{A} = (L, L_0, X, Z, E, \theta, \eta, C)$ *with the restriction that for all* $e = (l, \varphi, \phi, l') \in E$ *if* $\phi \neq \emptyset$ *then* φ *consists of atleast one atomic clock constraint* $x = c$ *for some* $x \in X, c \in \mathbb{N}$.

The resriction on the resets ensure that the fractional parts of all clocks remain the same at all points of time. For a clock valuation $\nu(x)$, let $frac(\nu(x))$ denote the fractional part of $\nu(x)$ and $int(\nu(x))$ denote the integral part. For example, if $\nu(x) = 7.02$, then $int(\nu(x)) = 7$ and $frac(\nu(x)) = .02$.

Example 1. The automaton $\mathcal{A}$ in the following figure is an IRTA.

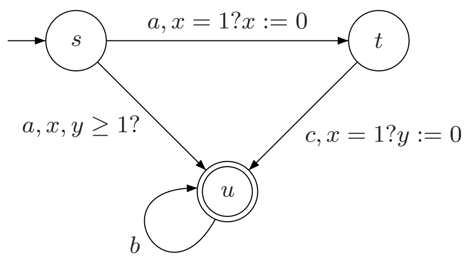


Fig. 3.0.1. An IRTA $\mathcal{A}$

Lemma 2. *Let* $\mathcal{A} = (L, L_0, \Sigma, X, E, F)$ *be an IRTA and* ν *be a clock valuation in any given run in* $\mathcal{A}$. *Then* $\forall x, y \in X$, $frac(\nu(x)) = frac(\nu(y))$. *[27]*

3.1 IRTA Regions

In this section, we look at the regions $\mathcal{R}$ of an IRTA. Given a set X of clocks, let $\mathcal{R}$ be a finite partitioning of $\mathbb{T}^X$. The notions successor of a region, compatibility with guards and compatibility with updates are same as mentioned in Section 2.1.

Let $c_m \in \mathbb{N}$ be the maximum constant occuring in the guards $\mathcal{G}(X)$ of the IRTA $\mathcal{A}$. For every clock $x \in X$, define a set of intervals $\mathcal{I}_x$, as

$$\mathcal{I}_x = \{[c] | 0 \leq c \leq c_m\} \cup \{(c, c+1) | 0 \leq c < c_m\} \cup \{(c_m, \infty)\}$$

Let α be a tuple $((I_x)_{x \in X}, \prec)$ where

1. $I_x \in \mathcal{I}_x$ and
2. $\prec$ is a total preorder on $X_0 = \{x \in X \mid I_x \text{ is of the form } (c, c+1)\}$.

The region associated with α is the set of valuations $\nu \in \mathbb{T}^X$ such that for all $x \in X$, $\nu(x) \in I_x$ and for all $x, y \in X_0$, $x \prec y$ iff $frac(\nu(x)) \leq frac(\nu(y))$. By Lemma 2, either $X_0 = X$ or $X_0 = \emptyset$. For all $x, y \in X$, $x \prec y$ and $y \prec x$. Thus we can safely consider α to be $\alpha = ((I_x)_{x \in X})$. The set of all such tuples α partitions $\mathbb{T}^X$ and this is the set we consider to be $\mathcal{R}$.

Definition 2 (Integral, Non-integral, Saturated region). *Let $\alpha = ((I_x)_{x\in X}) \in \mathcal{R}$ and let $X_m \subseteq X$ be such that $\forall x \in X_m, I_x = (c_m, \infty)$.*

- *α is said to be saturated if $X_m = X$*
- *else α is said to be integral if $\forall x \in X \setminus X_m$, I_x is of the form $[c]$*
- *else α is said to be non-integral if $\forall x \in X \setminus X_m$, I_x is of the form $(c, c+1)$*

Definition 3 (Immediate Successor). *For every clock region $\alpha \in \mathcal{R}$, its immediate successor α^i is defined as $\forall \nu \in \alpha, \forall t \in \mathbb{T}$ if $\nu + t \notin \alpha$ then $\exists t' \leq t$ such that $\nu + t' \in \alpha^i$.*

Lemma 3. *Every clock region $\alpha \in \mathcal{R}$ has a unique immediate successor $\alpha^i \in \mathcal{R}$.*

Proof. Let $\alpha = ((I_x)_{x\in X}) \in \mathcal{R}$. Let $X_m \subseteq X$ such that $\forall x \in X_m, I_x = (c_m, \infty)$. Then $\forall x \in X_m$, $I'_x = I_x = (c_m, \infty)$.

- If $X_m = X$ then $\alpha^i = \alpha$
- If α is non-integral i.e; $I_x = (c, c+1), \forall x \in X \setminus X_m$ and $X_m \subset X$ then $I'_x = [c+1]$
- If α is integral i.e; $I_x = [c], \forall x \in X \setminus X_m$ then $I'_x = (c, c+1)$ if $c < c_m$ else $I'_x = (c_m, \infty)$

Thus there exists a unique immediate successor for a given clock region. □

Note that the immediate successor of an integral clock region is a non-integral clock region and vice versa.

The following lemmas prove that $\mathcal{R}$ is indeed a set of regions and that it is compatible with the set of clock constraints and resets. The proof is as in [13].

Lemma 4. *Set $\mathcal{R}$ forms a set of regions.*

This lemma follows from the proof of Lemma 3.

Lemma 5. *The set of regions $\mathcal{R}$ is compatible with the set of clock constraints $\mathcal{G}(X)$.*

Lemma 6. *The set of regions $\mathcal{R}$ is compatible with the set $U_0(X)$ of clock resets.*

4 Clock Reduction and Decidability

In this section, we give a technique for reducing the number of clocks in a WIRTA to one. The clock reduction is due to Lemma 2. This simplifies the region partitioning to give regions which can be called *integral*, *non-integral* and *saturated.*

4.1 Untiming WIRTA

In this section, we look at some techniques for untiming a WIRTA $\mathcal{A}$. The spirit of these follow [27]. For simplicity, we consider WIRTAs where all the location invariants are *true*. The component η which assigns *true* to all locations will be omitted from the description of WIRTAs. All the results obtained can be extended to the case of having general location invariants.

Definition 4. *Let $\tau \in \mathbb{R}_+$, and let $int(\tau) = k$. Define*

$$dt(\tau) \triangleq \begin{cases} (\delta\checkmark)^k & \text{if } \tau \text{ is integral,} \\ (\delta\checkmark)^k\delta & \text{if } \tau \text{ is non-integral.} \end{cases}$$

Let $\tau_1 \leq \tau_2$ be two real numbers. $dte(\tau_1, \tau_2)$ is the $\delta\checkmark$-pattern that is to be right concatenated to $dt(\tau_1)$ to get $dt(\tau_2)$. □

For example, if $\tau_1 = 1.3$ and $\tau_2 = 2.9$, then $dt(\tau_1) = \delta\checkmark\delta$ while $dt(\tau_2) = \delta\checkmark\delta\checkmark\delta$. Therefore, $dte(\tau_1, \tau_2) = \checkmark\delta$.

Proposition 1. *Let $t_0t_1 \ldots t_n$ and $t'_0t'_1 \ldots t'_n$ be two sequences of time stamps such that $t_0 = t'_0 = 0$, and $dte(t_i, t_{i-1}) = dte(t'_i, t'_{i-1})$ for all $i \geq 1$. Then $int(t_i) = int(t'_i)$, and $frac(t_i) = 0$ iff $frac(t'_i) = 0$ for all i.*

In the following, we assume that all runs begin from time 0, and that time progresses in a weakly monotonic sense.

Definition 5. *Let $\Sigma = L \times \mathbb{T}$ where L is the set of locations of a given WIRTA. Define a language $\mathcal{W}$ over Σ as $\mathcal{W}=\{(l_0, t_0)(l_0, t_1)(l_1, t_1)(l_1, t_2)(l_2, t_2) \cdots (l_{n-1}, t_n)(l_n, t_n) \mid n \geq 1, t_i \geq t_{i-1}$ for $1 \leq i \leq n\}$. Let $f : \mathcal{W} \to (L \cup \{\delta, \checkmark\})^*$ be a function defined as $f(w) = l_0\ dte(t_1, t_0)\ l_1\ dte(t_2, t_1) l_2 \ldots l_{n-1}\ dte(t_n, t_{n-1}) l_n$.*

Let $w = l_0w_1l_1w_2 \ldots l_{n-1}w_nl_n$ be a word such that $\delta, \checkmark$ strictly alternate in each $w_i \in \{\delta, \checkmark\}^*$ as well as in $w_1 \ldots w_n$. Let $l_i \in L$.

Example 2. Let $w = (l_0, 0)(l_0, 0.5)(l_1, 0.5)(l_1, 1.2)(l_2, 1.2)(l_2, 1.5)(l_3, 1.5)(l_3, 2.1)(l_4, 2.1)$, $w' = (l_0, 0)(l_0, 0.2)(l_1, 0.2)(l_1, 1.7)(l_2, 1.7)(l_2, 1.9)(l_3, 1.9)(l_3, 2.3)(l_4, 2.3)$ and $w'' = (l_0, 2)(l_0, 2.7)(l_1, 2.7)(l_1, 3.2)(l_2, 3.2)(l_2, 3.3)(l_3, 3.3)(l_3, 4.05)(l_4, 4.05)$. Then $f(w) = f(w') = f(w'') = l_0\delta l_1\checkmark\delta l_2\epsilon l_3\checkmark\delta l_4$.

Definition 6. *Two words $w, w' \in \mathcal{W}$ are said to be f equivalent iff $f(w) = f(w')$.*

Let $t_0 = t'_0 = 0$. If $w = (l_0, t_0)(l_0, t_1)(l_1, t_1)(l_1, t_2)(l_2, t_2) \cdots (l_{n-1}, t_n)(l_n, t_n)$ and $w' = (l_0, t'_0)(l_0, t'_1)(l_1, t'_1)(l_1, t'_2)(l_2, t'_2) \cdots (l_{n-1}, t'_n)(l_n, t'_n)$ are f-equivalent, then by Proposition 1, $int(t_i) = int(t'_i)$ and $frac(t_i) = 0$ iff $frac(t'_i) = 0$.

Definition 7. *Let $\mathcal{A}$ be a WTA and let $l_0 \in L_0$. Consider two paths ρ and ρ' such that*

1. *$l_0 l_1 l_2 \cdots l_n$ is the sequence of locations constituting ρ such that l_i is visited at time t_i.*
2. *$l_0 l_1 l_2 \ldots l_n$ is the sequence of locations constituting ρ' such that l_i is visited at time t'_i.*

Define $g(\rho)$ to be $w = (l_0, t_0)(l_0, t_1)(l_1, t_1)(l_1, t_2) \ldots (l_{n-1}, t_{n-1})(l_{n-1}, t_n)(l_n, t_n)$. ($g(\rho')$ is $w' = (l_0, t'_0)(l_0, t'_1)(l_1, t'_1)(l_1, t'_2) \ldots (l_{n-1}, t'_{n-1})(l_{n-1}, t'_n)(l_n, t'_n)$). Then, ρ and ρ' are said to be equivalent ($\rho \cong \rho'$) iff $f(g(\rho)) = f(g(\rho'))$.

Proposition 2. *Let ρ and ρ' be two paths visiting the sequence of locations $l_0 l_1 \ldots l_n$ ($m_0 m_1 \ldots m_n$) in order, such that $l_i (m_i)$ is visited at at times $t_i (t'_i)$. Then $\rho \cong \rho'$ iff*

1. *$l_i = m_i$ for all i,*
2. *$dte(t_i, t_{i-1}) = dte(t'_i, t'_{i-1})$ for all $i \geq 1$.*

Proof. Let $\rho \cong \rho'$ be two paths visiting the sequence of locations $l_0 l_1 \ldots l_n$ ($m_0 m_1 \ldots m_n$) in order, such that location l_i (m_i) is visited at time $t_i (t'_i)$.

1. Since $\rho \cong \rho'$, we have $f(g(\rho)) = f(g(\rho')) = k_0 w_0 k_1 w_1 \ldots k_n$ where $w_i \in \{\delta, \checkmark\}^*$. Clearly, $l_i = m_i = k_i$ for all i.
2. Assume that j is the first index such that $dte(t_j, t_{j-1}) \neq dte(t'_j, t'_{j-1})$. Then $f(g(\rho))$ would differ from $f(g(\rho'))$, contradicting the assumption that $\rho \cong \rho'$.

Conversely, assume that for paths ρ, ρ' visiting locations $l_0 l_1 \ldots l_n$ ($m_0 m_1 \ldots m_n$) in order, at times $t_i (t'_i)$, we have $l_i = m_i$ and $dte(t_i, t_{i-1}) = dte(t'_i, t'_{i-1})$ for all $i \geq 1$. If $\rho \ncong \rho'$, then $f(g(\rho)) \neq f(g(\rho'))$. Since $l_i = m_i$, the only way they can differ is by $dte(t_i, t_{i-1})$ or $dte(t'_i, t'_{i-1})$ for some i. But this also is ruled out by the assumption. Hence, it must be that $\rho \cong \rho'$. □

Proposition 3. *Let $\mathcal{A}$ be a WIRTA. Let $\rho \cong \rho'$ be paths visiting the sequence of locations $l_0 l_1 \ldots l_n$ in order, such that l_i is visited at time t_i in ρ and at time t'_i in ρ', with $t_0 = t'_0 = 0$. Then ρ is a path in $\mathcal{T}_\mathcal{A}$ iff ρ' is a path in $\mathcal{T}_\mathcal{A}$.*

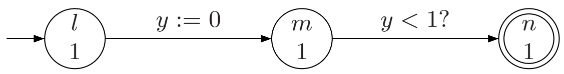
Proof. Let $\rho \cong \rho'$. We prove the result by induction on the lengths of paths ρ, ρ'. Let ρ, ρ' visit the sequence of states l_0, l_1 at times $t_0 = 0, t_1$ and $t'_0 = 0, t'_1$ respectively. Since $\rho \cong \rho'$, we have $dt(t_1) = dt(t'_1)$. This implies that $int(t_1) = int(t'_1)$ and $frac(t_1) = 0$ iff $frac(t'_1) = 0$. The valuation of any clock x, $\nu_1(x)$ after time elapse t_1 is $int(t_1) + frac(t_1)$, while the valuation $\alpha_1(x)$ after time elpase t'_1 is $int(t'_1) + frac(t'_1)$. Clearly, $int(\nu_1(x)) = int(\alpha_1(x))$ and $frac(\nu_1(x)) = 0$ iff $frac(\alpha_1(x)) = 0$. Thus, $\nu_1(x) \models \varphi_1$ for a constraint φ_1 iff $\alpha_1(x) \models \varphi_1$ for any clock $x \in X$ (from 5). Therefore, if $(l_0, \nu'_0, \mu'_0) \xrightarrow{t_1} (l_0, \nu_1, \mu_1) \xrightarrow{(\varphi_1, \phi_1)} (l_1, \nu'_1, \mu'_1)$ is a path in $\mathcal{T}_\mathcal{A}$, then so will be $(l_0, \nu'_0, \chi'_0) \xrightarrow{t'_1} (l_0, \alpha_1, \chi_1) \xrightarrow{(\varphi_1,\phi_1)} (l_1, \alpha'_1, \chi'_1)$, where the costs μ_i, χ_i are calculated in the usual way.

Assume that the result holds for paths of length $\leq j-1$. Now let $\rho \cong \rho'$ be paths of length j. Let r, r' be subpaths of ρ, ρ' obtained by visiting locations l_i, at times t_i (t'_i), $0 \leq i \leq j-1$. By the induction hypothesis, $r = (l_0, \nu'_0, \mu'_0) \xrightarrow{t_1} (l_0, \nu_1, \mu_1) \xrightarrow{(\varphi_1,\phi_1)} (l_1, \nu'_1, \mu'_1) \xrightarrow{t_2} (l_1, \nu_2, \mu_2) \xrightarrow{(\varphi_2,\phi_2)} (l_2, \nu'_2, \mu'_2) \cdots (l_{j-2}, \nu_{j-1}, \mu_{j-1}) \xrightarrow{(\varphi_{j-1},\phi_{j-1})} (l_{j-1}, \nu'_{j-1}, \mu'_{j-1})$ is a path in $\mathcal{T}_\mathcal{A}$ iff $r' = (l_0, \nu'_0, \chi'_0) \xrightarrow{t'_1} (l_0, \alpha_1, \chi_1) \xrightarrow{(\varphi_1,\phi_1)} (l_1, \alpha'_1, \chi'_1) \xrightarrow{t'_2} (l_1, \alpha_2, \chi_2) \xrightarrow{(\varphi_2,\phi_2)} (l_2, \alpha'_2, \chi'_2) \cdots (l_{j-2}, \alpha_{j-1}, \chi_{j-1}) \xrightarrow{(\varphi_{j-1},\phi_{j-1})} (l_{j-1}, \alpha'_{j-1}, \chi'_{j-1})$ is.

Since location l_j is visited at time t_j, for any clock x, $\nu_j(x) = t_j - t_k$ where k is the largest index less than j in r when x was reset (t_k is integral), or $t_k = 0$ in which case x was never reset. Similarly, $\alpha_j(x) = t'_j - t'_k$ with t'_k integral or $\alpha_j(x) = t'_j$ for r'. Hence, $\nu_j(x) = int(t_j) + frac(t_j) - t_k$, while $\alpha_j(x) = int(t'_j) + frac(t'_j) - t'_k$. Since $dte(t_j, t_{j-1}) = dte(t'_j, t'_{j-1})$ for all $j \geq 1$, we have $dte(t_j, t_k) = dte(t'_j, t'_k)$. This implies that $int(t_j - t_k) = int(t'_j - t'_k)$ and $frac(t_j - t_k) = 0$ iff $frac(t'_j - t'_k) = 0$ (from Proposition 1). Hence, $\nu_j(x) \models \varphi_j$ iff $\alpha_j(x) \models \varphi_j$ for any constraint φ_j, $x \in X$. Thus, we can extend r, r' by the transitions $(l_{j-1}, \nu'_{j-1}, \mu'_{j-1}) \xrightarrow{t_j} (l_{j-1}, \nu_j, \mu_j) \xrightarrow{(\varphi_j,\phi_j)} (l_j, \nu'_j, \mu'_j)$ (correspondingly, $(l_{j-1}, \alpha'_{j-1}, \chi'_{j-1}) \xrightarrow{t'_j} (l_{j-1}, \alpha_j, \chi_j) \xrightarrow{(\varphi_j,\phi_j)} (l_j, \alpha'_j, \chi'_j)$) so that the former is a path in $\mathcal{T}_\mathcal{A}$ iff the latter is. □

Corollary 1. *The above result is not true if $\mathcal{A}$ is a WTA but not a WIRTA.*

Proof. Consider the following WTA.



Consider paths ρ, ρ' obtained by visiting the sequence of states l, m, n at times $t_0 = 0, t_1 = 0.3, t_2 = 1.2$ and $t'_0 = 0, t'_1 = 0.3, t'_2 = 1.5$ respectively. Clearly, $dte(t_i, t_{i-1}) = dte(t'_i, t'_{i-1})$ for $i = 1, 2$. $f(g(\rho)) = f(g(\rho')) = l\delta m\checkmark\delta n$. Hence, $\rho \cong \rho'$.

Clearly, $\rho = (l, 0, 0) \xrightarrow{0.3} (l, 0.3, 0.3) \xrightarrow{y:=0} (m, 0, 0.3) \xrightarrow{1.2} (m, 0.9, 1.2) \xrightarrow{y<1?} (n, 0.9, 1.2)$ is a path in the automaton. However, there is no path of the form $\rho' = (l, 0, 0) \xrightarrow{0.3} (l, 0.3, 0.3) \xrightarrow{y:=0} (m, 0, 0.3) \xrightarrow{1.5} (m, 1.2, 1.5) \longrightarrow (n, 1.2, 1.5)$ in the automaton. □

As a first step in obtaining a one clock WIRTA from $\mathcal{A}$, we construct from $\mathcal{A}$, a marked automaton $\mathcal{M}_\mathcal{A}$. The time elapse between integral and non-integral regions in $\mathcal{A}$ is marked in $\mathcal{M}_\mathcal{A}$ using symbols δ and $\checkmark$. The locations we use in $\mathcal{M}_\mathcal{A}$ are of the form (l, α) where l is a location of L, and α is a region for the clocks $X \cup \{n\}$, where n is a new clock that is reset only when a clock reset happens in $\mathcal{A}$. A time elapse transition in $\mathcal{A}$ of the form $(l, \nu) \xrightarrow{t} (l, \nu + t)$ where

$\nu(x)$ is integral and $\nu(x)+t$ is non-integral for $x \in X$ is represented in $\mathcal{M}_{\mathcal{A}}$ as a discrete transition $(l,\alpha) \xrightarrow{\delta} (l,\alpha^i)$ where α^i is the immediate successor of α. Similarly, a time elapse transition in $\mathcal{A}$ of the form $(l,\nu) \xrightarrow{t} (l,\nu+t)$ where $\nu(x)$ is non-integral and $\nu(x)+t$ is integral for $x \in X$ is represented in $\mathcal{M}_{\mathcal{A}}$ as a discrete transition $(l,\alpha) \xrightarrow{\checkmark} (l,\alpha^i)$. $\mathcal{M}_{\mathcal{A}}$ has an extra clock f that keeps track of the progress of time between integral and non-integral regions. A guard $f \in (0,1)$ should be satisfied on a δ transition, while a guard $f=1$ should be satisfied on a $\checkmark$ transition. f is reset to zero everytime it attains the value of 1 on the $\checkmark$ transition. A discrete transition (l,φ,ϕ,l') of $\mathcal{A}$ is represented by marked discrete transitions from (l,α) to (l',α') on ϵ such that $\alpha \models \varphi$, $f=0$ if α is integral, and $f \in (0,1)$ if α is non-integral, and α' is obtained from α by resetting the clocks in ϕ (if any) and n. If $\phi = \emptyset$, then $\alpha' = \alpha$.

The only possible time elapse transitions in $\mathcal{M}_{\mathcal{A}}$ are of the form $((l,\alpha),\zeta) \xrightarrow{t} ((l,\alpha),\zeta+t)$ where $\zeta(f)$ is the valuation of clock f. Due to the guards on the $\delta, \checkmark, \epsilon$ transitions, the possible time t elapsed should satisfy one of the following:

1. $\zeta(f)=0, 0<\zeta(f)+t<1$,
2. $0<\zeta(f), \zeta(f)+t<1$,
3. $0<\zeta(f)<1, \zeta(f)+t=1$.

We now give the formal definition of $\mathcal{M}_{\mathcal{A}}$.

Definition 8. *Given a WIRTA* $\mathcal{A}=(L,L_0,X,Z,E,\theta,C)$*, we construct a marked weighted timed automaton* $\mathcal{M}_{\mathcal{A}}$ *corresponding to* $\mathcal{A}$ *as* $\mathcal{M}_{\mathcal{A}} = (Q,Q_0,\{f\},Z,E_m,\theta_m,C_m)$ *where*

- $Q = L \times \mathcal{R}$ *where* $\mathcal{R}$ *is the set of regions defined for* $X \cup \{n\}$ *where* $n \notin X$,
- $Q_0 = L_0 \times \{\alpha_0\}$ *such that* $\alpha_0 = \{\nu_0\}, \nu_0(x)=0$ *for all* $x \in X \cup \{n\}$,
- Z *is the set of costs as in* $\mathcal{A}$,
- $E_m \subseteq Q \times \{\delta,\checkmark,\epsilon\} \times \mathcal{G}(\{f\}) \times U_0(\{f\}) \times Q$ *is the set of edges. For* $q=(l,\alpha)$ *and* $q'=(l',\alpha')$*, an edge* $e_m=(q,a,\varphi_m,\phi_m,q') \in E_m$ *is such that*
  1. *if* $\alpha(x)=(c_m,\infty)$ *for all* $x \in X \cup \{n\}$*, then* $q=q'$, $a \in \{\delta,\checkmark\}$, $\varphi_m ::$ *true and* $\phi_m = \phi$,
  2. *if* $l=l'$, α *is integral and* $\alpha'=\alpha^i$*, then* $a=\delta$, $\varphi_m :: 0<f<1$ *and* $\phi_m=\emptyset$,
  3. *if* $l=l'$, α' *is integral and* $\alpha'=\alpha^i$*, then* $a=\checkmark$, $\varphi_m :: f=1$ *and* $\phi_m=\{f\}$,
  4. *For a discrete transition* $(l,\varphi,\phi,l') \in E$*, there exists a transition*$((l,\alpha),\epsilon,\varphi_m,\emptyset,(l',\alpha')) \in E_m$ *such that* *(1)* $\alpha \models \varphi$, *(2)* $\alpha' = \alpha[\phi \cup \{n\}]$ *if* $\phi \neq \emptyset$*, else* $\alpha'=\alpha$*, and (3)* $\varphi_m :: f=0$ *if* α *is integral, else* $\varphi_m :: 0<f<1$,
- $\theta_m : Q \to 2^{\Sigma}$ *such that* $\theta_m(q)=\theta(l)$ *for* $q=(l,\alpha)$,
- $C_m : Q \cup E_m \to \mathbb{N}^{|Z|}$ *such that*
  1. $C_m(q)=C(l)$ *if* $q=(l,\alpha)$,
  2. $C_m(e_m)=C(e)$ *if* $e_m=(q,\epsilon,\varphi_m,\phi_m,q')$, $e=(l,\varphi,\phi,l')$, $q=(l,\alpha)$ *and* $q'=(l',\alpha')$,

3. $C_m(e_m) = 0$ *if* $e_m = (q, \delta, \varphi_m, \phi_m, q')$ *or* $e_m = (q, \checkmark, \varphi_m, \phi_m, q')$ *where* $q = (l, \alpha)$ *and* $q' = (l, \alpha')$.

The semantics of $\mathcal{M}_A$=$(Q, Q_0, Z, E_m, \theta_m, C_m)$ is given by a labelled timed transition system $\mathcal{T}_\mathcal{M}$ just as in the case of $\mathcal{A}$.

A path is a finite(infinite) sequence of consecutive transitions in the transition system $\mathcal{T}_\mathcal{M}$. A path ρ_m starting at $((l_0, \alpha_0), \gamma_0, \chi_0)$ with $\gamma_0 = 0$ and $\chi_0(z) = 0, \forall z \in Z$ is denoted as $r = ((l_0, \alpha_0), \gamma_0, \chi_0) \xrightarrow{t_{1,1}} ((l_0, \alpha_0), \gamma_1, \chi_1) \xrightarrow{\delta} ((l_0, \alpha_1), \gamma_1, \chi_1) \xrightarrow{t_{1,2}} ((l_0, \alpha_1), \gamma_2, \chi_2) \xrightarrow{\checkmark} ((l_0, \alpha_2), 0, \chi_2) \ldots \xrightarrow{t_{1,k+1}} ((l_0, \alpha_k), \gamma_{k+1}, \chi_{k+1}) \xrightarrow{a} ((l_0, \alpha_{k+1}), \gamma'_{k+1}, \chi_{k+1}) \xrightarrow{\epsilon} ((l_1, \alpha'_{k+1}), \gamma'_{k+1}, \chi'_{k+1}) \ldots \xrightarrow{\epsilon} ((l_n, \alpha'_m), \gamma'_m, \chi'_m)$, where $a = \delta$ iff $0 < \gamma_{k+1} < 1$ and $a = \checkmark$ iff $\gamma'_{k+1} = 0$. Note that the value of χ_j and γ_j in a state $((l_i, \alpha_{j-1}), \gamma_j, \chi_j)$ are given by $\chi_j = \chi_{j-1} + C_m(l_i, \alpha_{j-1}) * (t_{i+1,j} - t_{i+1,j-1})$ and $\gamma_j = \gamma_{j-1} + t_{i+1,j} - t_{i+1,j-1}$. The values of χ'_j and γ'_j in $((l_{i+1}, \alpha'_j), \gamma'_j, \chi'_j)$ after a transition $((l_i, \alpha_j), \gamma_j, \chi_j) \xrightarrow{\epsilon} ((l_{i+1}, \alpha'_j), \gamma'_j, \chi'_j)$ are given by $\chi'_j = \chi_j + C_m((l_i, \alpha_j), \epsilon, (l_{i+1}, \alpha'_j))$ and $\gamma'_j = \gamma_j$. γ_j does not change due to δ or ϵ transitions while after a $\checkmark$, it is zero as f is reset.

Note: It should be noted that due to the loop consisting of both δ and $\checkmark$ over locations with regions α such that $\alpha(x) = (c_m, \infty)$ for all x, the paths might contain arbitrary sequences of $\delta, \checkmark$. We restrict $\mathcal{T}_\mathcal{M}$ to only those paths where $\delta, \checkmark$ strictly alternate.

Lemma 7. *For every path ρ from $((l_0, \alpha_0), \gamma_0, \chi_0)$ to $((l_n, \alpha'_m), \gamma'_m, \chi'_m)$ in $\mathcal{T}_M$, there exists an equivalent path ρ' from $((l_0, \alpha_0), \gamma_0, \chi_0)$ to $((l_n, \alpha'_m), \gamma'_m, \chi'_m)$ in which all the ϵ transitions immediately follow a δ or $\checkmark$ transition (without any time elapse in between).*

Proof. By construction of $\mathcal{M}_\mathcal{A}$, the guard $f = 0$ or $0 < f < 1$ must be satisfied for an ϵ transition. It is clear that $f = 0$ only when it gets reset by a $\checkmark$ transition. Hence, for the case $f = 0$, an ϵ transition happens immediately after a $\checkmark$ transition.

Lets now consider the case $0 < f < 1$ while taking an ϵ transition. Consider a sequence ρ_p of transitions in ρ in which there is time elpase between a δ transition and the succeeding ϵ transition. Let $\rho_p = \xrightarrow{a} ((l_i, \alpha_j), 0, \chi_j) \xrightarrow{t_{i,j+1}} ((l_i, \alpha_j), \gamma_{j+1}, \chi_{j+1}) \xrightarrow{\delta} ((l_i, \alpha_{j+1}), \gamma_{j+1}, \chi_{j+1}) \xrightarrow{t_{i,j+2}} ((l_i, \alpha_{j+1}), \gamma_{j+2}, \chi_{j+2}) \xrightarrow{\epsilon} ((l_{i+1}, \alpha'_{j+1}), \gamma'_{j+2}, \chi'_{j+2})$, $a \in \{\epsilon, \checkmark\}$. Note that $\gamma'_{j+2} = t_{i,j+1} + t_{i,j+2}$ and $\chi_{j+2} = \chi_j + k_1$ and $\chi'_{j+2} = \chi_{j+2} + k_2$ where k_1, k_2 are the costs accumulated due to time elpase $t_{i,j+1} + t_{i,j+2}$ in l_i and the ϵ transition respectively.

Since the ϵ transition satisfies $0 < f < 1$, we know that $0 < t_{i,j+1} + t_{i,j+2} < 1$. It is easy to see that the sequence ρ'_p of transitions given by $\rho'_p = \xrightarrow{a} ((l_i, \alpha_j), 0, \chi_j) \xrightarrow{t_{i,k+1}} ((l_i, \alpha_j), \gamma_{k+1}, \chi_{k+1}) \xrightarrow{\delta} ((l_i, \alpha_{k+1}), \gamma_{k+1}, \chi_{k+1}) \xrightarrow{\epsilon} ((l_{i+1}, \alpha'_{k+1}), \gamma'_{k+1}, \chi'_{k+1})$ can be taken in place of ρ_p with $t_{i,k+1} = t_{i,j+1} + t_{i,j+2}$.

The δ transition only demands that the guard $0 < f < 1$ holds good - this is clearly satisfied. Further, by the construction of $\mathcal{M}_{\mathcal{A}}$, we have $\gamma'_{k+1} = t_{i,j+1} + t_{i,j+2} = \gamma'_{j+2}$, $\chi_{k+1} = \chi_j + k_1$ and $\chi'_{k+1} = \chi_{k+1} + k_2$.

□

Henceforth, we consider only those paths in which all the ϵ transitions immediately follow the δ or ✓ transitions.

Let ρ be a path in $\mathcal{T}_{\mathcal{M}}$. Let ρ' be a subpath of ρ from (l,α) to (l,α_{k+1}) having one of the following forms:

1. ρ' is a prefix of ρ from (l,α) to (l,α_{k+1}) ((l,α) is the initial location) such that (l,α_{k+1}) is the source of an ϵ transition in ρ,
2. ρ' is a subpath of ρ of the form $\xrightarrow{\epsilon} ((l,\alpha),\gamma,\chi) \xrightarrow{t_{1,1}} ((l,\alpha),\gamma_1,\chi_1) \xrightarrow{\delta} ((l,\alpha_1),\gamma_1,\chi_1) \xrightarrow{t_{1,2}} ((l,\alpha_1),\gamma_2,\chi_2) \xrightarrow{\checkmark} ((l,\alpha_2),0,\chi_2)\ldots \xrightarrow{t_{1,k+1}} ((l,\alpha_k),\gamma_{k+1},\chi_{k+1}) \xrightarrow{a} ((l,\alpha_{k+1}),\gamma_{k+1},\chi_{k+1}) \xrightarrow{\epsilon} ((l',\beta),\gamma'_{k+1},\chi'_{k+1})$, $a \in \{\delta,\checkmark\}$.

Note that there are no ϵ transitions in the path between (l,α) to (l,α_{k+1}). Let $w \in \{\delta,\checkmark\}^*$ be a word obtained by concatenating the δ's and ✓'s appearing on the edges between the two locations $(l,\alpha),(l,\alpha_{k+1})$. Let $h(\rho)$ denote the word $l_0 w_1 l_1 w_1 \ldots l_{n-1} w_n l_n$ where w_{i+1} is the word over $\{\delta,\checkmark\}$ between (l_i,α) and (l_i,α_{k_i+1}) in subpaths ρ' of ρ of the above mentioned form.

Proposition 4. *Let (l,α) and (l,α') be two locations in $\mathcal{M}_{\mathcal{A}}$. Let (l,α') be reachable in $\mathcal{M}_{\mathcal{A}}$ from (l,α) by a sequence of time elapse and $\delta,\checkmark$ transitions. Then for a word $w \in \{\delta,\checkmark\}^*$ leading (l,α) to (l,α'), we have*

1. *$\delta,\checkmark$ strictly alternate in w,*
2. *$w = dte(t',t)$ such that $t \in \alpha(n), t' \in \alpha'(n)$.*

Proof. Let (l,α') be reachable in $\mathcal{M}_{\mathcal{A}}$ from (l,α) on reading a word $w \in \{\delta,\checkmark\}^*$. Let $\alpha(n) = [c]$ and $\alpha'(n) = (c+k, c+k+1)$, $k \geq 0$. By construction of $\mathcal{M}_{\mathcal{A}}$, we have the sequence of transitions $(l,\alpha) \xrightarrow{\delta} (l,\alpha_1) \xrightarrow{\checkmark} \ldots \xrightarrow{\delta} (l,\alpha')$ such that if a δ transition is taken then n becomes non-integral, and if a ✓ transition is taken then n becomes integral. Since $\alpha'(n) = (c+k, c+k+1)$ and $\alpha(n) = [c]$, it is clear that the number of ✓'s seen is k and the number of δ's seen is $k+1$. It is easy to see that 1,2 hold good.

A similar argument can be given when $\alpha(n) \in \mathcal{I}_x \setminus \{c_m, \infty\}$ □

Definition 9 (Consistency). *Given a WIRTA $\mathcal{A}$ and its marked automaton $\mathcal{M}_{\mathcal{A}}$, two paths $\rho \in \mathcal{T}_{\mathcal{A}}$ and $\rho' \in \mathcal{T}_{\mathcal{M}}$ are said to be consistent iff $f(g(\rho)) = h(\rho')$.*

Lemma 8. *Let $\mathcal{A}$=$(L, L_0 X, Z, E, \theta, C)$ be a WIRTA and let $\mathcal{M}_{\mathcal{A}}$=$(Q, Q_0, \{f\}, Z, E_m, \theta_m, C_m)$ be its marked automaton.*

1. *For every path ρ of $\mathcal{T}_{\mathcal{A}}$, there exists a path ρ_m of $\mathcal{T}_{\mathcal{M}}$ such that ρ,ρ_m are consistent.*

2. *For every path* ρ_m *of* $\mathcal{T}_\mathcal{M}$ *where the* $\delta, \checkmark$ *strictly alternate, there exists a path* ρ *of* $\mathcal{T}_\mathcal{A}$ *such that* ρ, ρ_m *are consistent.*
3. *Let* ρ *be a path in* $\mathcal{T}_\mathcal{A}$ *consistent with a path* ρ' *in* $\mathcal{T}_\mathcal{M}$. *Then all paths* ρ'' *in* $\mathcal{T}_\mathcal{A}$ *such that* $\rho'' \cong \rho$ *will be consistent with* ρ'.

Proof. 1. We prove by induction on the lengths of paths. For the base case, consider a path $\rho = (l_0, \nu'_0, \mu'_0) \xrightarrow{t_1} (l_0, \nu_1, \mu_1) \xrightarrow{(\varphi_1, \phi_1)} (l_1, \nu'_1, \mu'_1)$ in $\mathcal{T}_\mathcal{A}$. Let $int(t_1) = k$. Then by construction of $\mathcal{M}_\mathcal{A}$, corresponding to $(l_0, \nu'_0, \mu'_0) \xrightarrow{t_1} (l_0, \nu_1, \mu_1)$, there exists a path $r = ((l_0, \alpha'_0), \gamma'_0, \chi'_0) \xrightarrow{t_{1,1}} ((l_0, \alpha'_0), \gamma'_1, \chi'_1) \xrightarrow{\delta} ((l_0, \alpha'_1), \gamma'_1, \chi'_1) \xrightarrow{t_{1,2}} ((l_0, \alpha'_1), \gamma'_2, \chi'_2) \xrightarrow{\checkmark} ((l_0, \alpha'_2), 0, \chi'_2) \dots \xrightarrow{t_{1,2k+1}} ((l_0, \alpha'_{2k}), \gamma'_{2k+1}, \chi'_{2k+1}) \xrightarrow{\delta} ((l_0, \alpha'_{2k+1}), \gamma'_{2k+1}, \chi'_{2k+1})$ if $frac(t_1) \neq 0$,

[or $r = ((l_0, \alpha'_0), \gamma'_0, \chi'_0) \xrightarrow{t_{1,1}} ((l_0, \alpha'_0), \gamma'_1, \chi'_1) \xrightarrow{\delta} ((l_0, \alpha'_1), \gamma'_1, \chi'_1) \xrightarrow{t_{1,2}} ((l_0, \alpha'_1), \gamma'_2, \chi'_2) \xrightarrow{\checkmark} ((l_0, \alpha'_2), 0, \chi'_2) \dots \xrightarrow{t_{1,2k}} ((l_0, \alpha'_{2k-1}), \gamma'_{2k}, \chi'_{2k}) \xrightarrow{\checkmark} ((l_0, \alpha'_{2k}), 0, \chi'_{2k})$ if $frac(t_1) = 0$] such that

(a) $0 < t_{i,j} < 1, 1 \leq j \leq 2k+1$, $t_1 = t_{1,2k+1}$ ($t_{1,2k}$ if $frac(t_1) = 0$),
(b) The region $\alpha'_{2i+1} = ((I_x)_{x \in X \cup \{n\}})$ is such that $I_x = (i, i+1)$ whenever $\alpha'_{2i} = (I'_x)_{x \in X \cup \{n\}}$ with $I'_x = [i]$ for $i \geq 0$ and $\alpha'_{2i} = ((I_x)_{x \in X \cup \{n\}})$ is such that $I_x = [i+1]$ whenever $\alpha'_{2i-1} = (I'_x)_{x \in X \cup \{n\}}$ with $I'_x = (i, i+1)$ for $i \geq 1$,
(c) The cost $\chi'_i = \chi'_{i-1} + C(l_0) * (t_{1,i} - t_{1,i-1})$ for all i. Since $t_1 = t_{1,2k+1}$, we have $\chi'_{2k+1} = \mu_1$ (respectively, $\chi'_{2k} = \mu_1$ in the case $frac(t_1) = 0$),
(d) $\gamma'_i = \gamma'_{i-1} + t_{1,i} - t_{1,i-1}$ or $\gamma'_i = 0 + t_{1,i} - t_{1,i-1}$, for all i,
(e) The number of $\checkmark$s in r is k. The number of δs is k if $frac(t_1) = 0$ and is $k+1$ if $frac(t_1) \neq 0$.
(f) $\nu_1(x) \in \alpha'_{2k+1}(x)$ or $\nu_1(x) \in \alpha'_{2k}(x)$ for all $x \in X$ depending on whether $frac(t_1)$ is non-zero or not.

From the above, it is clear that $\alpha'_{2k+1} \models \varphi_1$ ($\alpha'_{2k} \models \varphi_1$) iff $\nu_1(x) \models \varphi_1$ for all $x \in X$. Hence, corresponding to ρ, we have in $\mathcal{T}_\mathcal{M}$ the path ρ' obtained from r by adding the transition on ϵ from the last state of r as $((l_0, \alpha'_{2k+1}), \gamma'_{2k+1}, \chi'_{2k+1}) \xrightarrow{\epsilon} ((l_1, \alpha_{2k+2}), \gamma_{2k+2}, \chi_{2k+2})$ if $frac(t_1) \neq 0$ or $((l_0, \alpha'_{2k}), \gamma'_{2k}, \chi'_{2k}) \xrightarrow{\epsilon} ((l_1, \alpha_{2k+1}), \gamma_{2k+1}, \chi_{2k+1})$ if $frac(t_1) = 0$. Here, $\alpha_{2k+2} = \alpha'_{2k+1}[\phi_1 \cup \{n\}]$ if $\phi_1 \neq \emptyset$, (otherwise, $\alpha'_{2k+1} = \alpha_{2k+2}$), $\gamma'_{2k+1} = \gamma_{2k+2}$ and $\chi_{2k+2} = \chi'_{2k+1} + C(e) = \mu'_1$ where $e = (l_0, \varphi_1, \phi_1, l_1)$.

[Similarly, $\alpha_{2k+1} = \alpha'_{2k}[\phi_1 \cup \{n\}]$ if $\phi_1 \neq \emptyset$, else, $\alpha_{2k+1} = \alpha'_{2k}$, $\gamma_{2k+1} = \gamma'_{2k}$ and $\chi_{2k+1} = \chi'_{2k} + C(e) = \mu'_1$ where $e = (l_0, \varphi_1, \phi_1, l_1)$].

It is clear that $g(\rho) = (l_0, t_0)(l_0, t_1)(l_1, t_1)$, $f(g(\rho)) = l_0 w l_1$ where $w = dte(t_1, t_0)$. Also, $h(\rho') = l_0 w' l_1$ where w' is the word over $\{\delta, \checkmark\}$ leading from (l_0, α'_0) to (l_0, α'_{2k+1}) (or from (l_0, α'_0) to (l_0, α'_{2k}) when $frac(t_1) = 0$). By Proposition 4, we have $w = dte(t'_1, t'_0)$ where $t'_1 \in \alpha'_{2k+1}(n), t'_0 \in \alpha'_0(n) = 0$. Since we know that $t_1 = \nu_1(x) \in \alpha'_{2k+1}(x)$ and since $\alpha'_{2k+1}(n) = \alpha'_{2k+1}(x)$ for all $x \in X$, we have $w' = dte(t'_1, t'_0) = dte(t_1, t_0) = w$. Hence, ρ, ρ' are consistent.

Assume that for every path ρ of length $\leq j-1$ in $\mathcal{T}_{\mathcal{A}}$, we have a path ρ' in $\mathcal{T}_{\mathcal{M}}$ consistent with ρ. Consider a path ρ of length j in $\mathcal{T}_{\mathcal{A}}$.
Let $\rho = (l_0, \nu'_0, \mu'_0) \xrightarrow{t_1} (l_0, \nu_1, \mu_1) \xrightarrow{(\varphi_1,\phi_1)} (l_1, \nu'_1, \mu'_1) \xrightarrow{t_2} (l_1, \nu_2, \mu_2) \xrightarrow{(\varphi_2,\phi_2)} (l_2, \nu'_2, \mu'_2) \cdots (l_{j-2}, \nu'_{j-2}, \mu'_{j-2}) \xrightarrow{t_{j-1}} (l_{j-2}, \nu_{j-1}, \mu_{j-1}) \xrightarrow{(\varphi_{j-1},\phi_{j-1})} (l_{j-1}, \nu'_{j-1}, \mu'_{j-1}) \xrightarrow{t_j} (l_{j-1}, \nu_j, \mu_j) \xrightarrow{(\varphi_j,\phi_j)} (l_j, \nu'_j, \mu'_j)$.
Breaking ρ into paths ρ_1 and ρ_2 of lengths $j-1$ and one and using the inductive hypothesis, it is easy to find paths ρ'_1, ρ'_2 in $\mathcal{M}_{\mathcal{A}}$ consistent with ρ_1, ρ_2 such that the accumulated costs in the locations of ρ_1, ρ_2 agree with those of ρ'_1, ρ'_2.
The path ρ' obtained by joining ρ'_1, ρ'_2 in $\mathcal{M}_{\mathcal{A}}$ is a path consistent with ρ.
2. Similar to 1.
3. Let ρ be a path in $\mathcal{T}_{\mathcal{A}}$ consistent with the path ρ' in $\mathcal{T}_{\mathcal{M}}$. For any path ρ'' in $\mathcal{T}_{\mathcal{A}}$ such that $\rho \cong \rho''$, we have $f(g(\rho)) = f(g(\rho''))$. Since $f(g(\rho)) = h(\rho')$, we also have $f(g(\rho'')) = h(\rho')$. Hence, ρ'' is also consistent with ρ'.

□

Corollary 2. *In the case of a WTA which is not a WIRTA, the 3rd property listed in the above theorem need not hold good due to Corollary 1.*

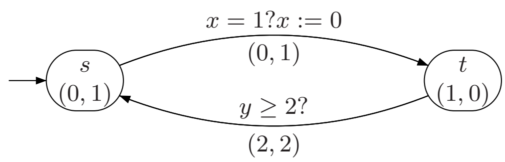


Fig. 4.1.1. W-IRTA $\mathcal{A}$.

4.2 One clock WIRTA

In this section, we give a construction to obtain a single clock $WIRTA$ $\mathcal{A}'$ from $\mathcal{M}_{\mathcal{A}}$ such that for every run $\rho \in \mathcal{A}'$, there exists a run $\rho' \in \mathcal{A}$ preserving the costs.

Definition 10 ($\delta - \checkmark$ sequence). *Let (l, α) be a location in $\mathcal{M}_{\mathcal{A}}$. A $\delta - \checkmark$ sequence starting at (l, α) is a sequence of locations $l_\alpha = (l, \alpha_0)(l, \alpha_1) \ldots (l, \alpha_n)$ such that $\alpha_0 = \alpha$ and $\forall j \geq 0$, $\alpha_{j+1} = \alpha_j^i$ such that any path in $\mathcal{T}_{\mathcal{M}}$ consisting of only these locations is of the form $((l, \alpha_0), \gamma_0, \chi_0) \xrightarrow{t_{1,1}} ((l, \alpha_0), \gamma_1, \chi_1) \xrightarrow{a} ((l, \alpha_1), \gamma_1, \chi_1) \xrightarrow{t_{1,2}} ((l, \alpha_1), \gamma_2, \chi_2) \xrightarrow{a'} ((l, \alpha_2), \gamma_2, \chi_2) \ldots \xrightarrow{t_{1,k+1}} ((l, \alpha_k), \gamma_{k+1}, \chi_{k+1}) \xrightarrow{\delta} ((l, \alpha_{k+1}), \gamma_{k+1}, \chi_{k+1})$ where*

1. *$a, a' \in \{\delta, \checkmark\}$,*

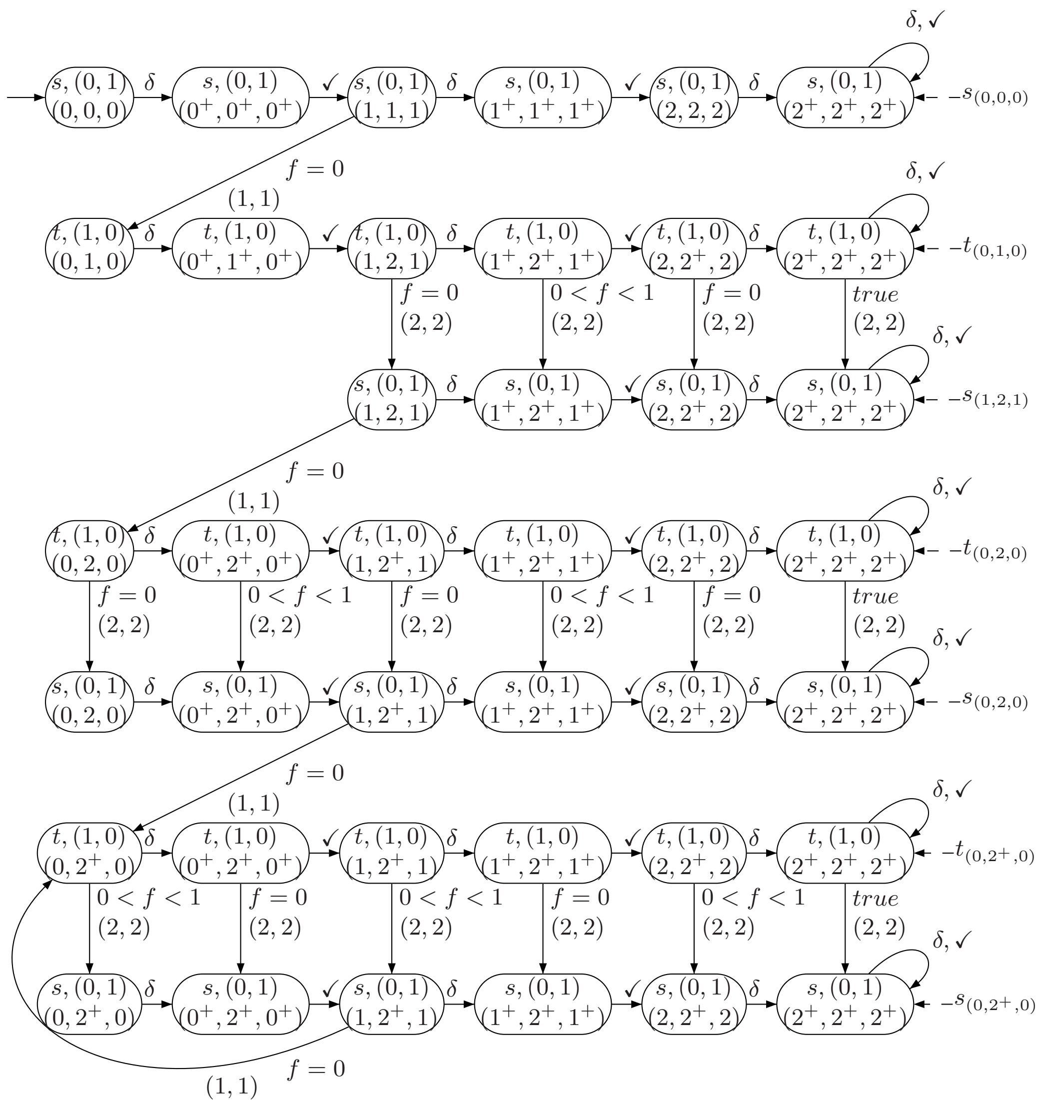


Fig. 4.1.2. Marked Weighted Automaton $\mathcal{M}$ corresponding to the W-IRTA of Figure 4.1.1. Locations of $\mathcal{M}$ are of the form (l, α, r) Where l is a location in $\mathcal{A}$, $\alpha = (I_x, I_y, I_n) \in \mathcal{R}$ the set of regions for the set $X \cup \{n\}$ and $r \in \mathbf{N}^{|Z|}$ is the cost of the location. Intervals of α are represented as follows: c for $[c]$, c^+ for $(c, c+1)$ if $c < c_m$ (here c_m =2) else c^+ stands for (c_m, ∞). Also note that for readability, some of the states have been replicated and the resets and guards of f associated with δ and ✓ edges are not shown. $s_{(0,0,0)}$ stands for the $\delta - ✓$ sequence starting at location $(s, (0, 0, 0))$.

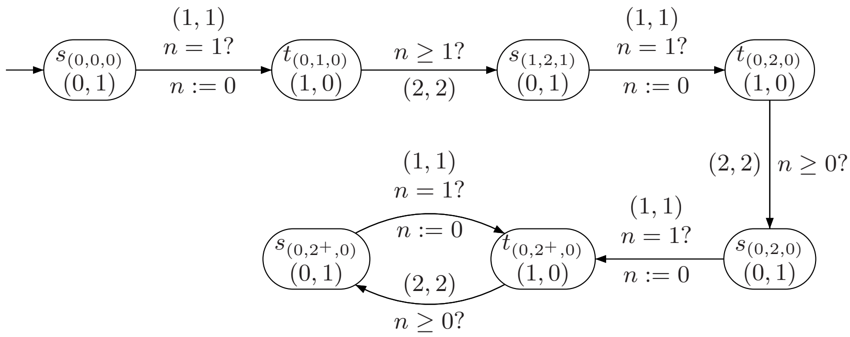


Fig. 4.1.3. Single clock W-IRTA $\mathcal{A}'$ corresponding to MWA $\mathcal{M}$ of Figure 4.1.

2. *The $\delta, \checkmark$ moves strictly alternate,*
3. *The location (l, α_{k+1}) has a self loop on $\delta, \checkmark$ in $\mathcal{M}_{\mathcal{A}}$.*

Let $L' = \{l_\alpha \mid (l, \alpha) \in L \times \mathcal{R}\}$. The one clock WIRTA is built by considering L' as the set of locations. Recall that the locations of $\mathcal{M}_{\mathcal{A}}$ were of the form (l, α) where α was a region for $X \cup \{n\}$. $\alpha(n)$ was made zero after an ϵ transition only whenever some clock $x \in X$ was reset as part of the transition in $\mathcal{A}$. Thus, the clock n records the time elpased between 2 reset transitions of $\mathcal{A}$. Here, we use this clock n as the only clock of $\mathcal{A}'$. The initial location will be s_{α_0}, where $s \in L_0$ and α_0 is the region $(0, 0, \ldots, 0)$. Treating a sequence l_α as a location, we can enable transitions from l_α whenever there was a transition from (l, α_i) in $\mathcal{M}_{\mathcal{A}}$. The guard to be satisfied on such a transition is that the value of n agrees with $\alpha_i(n)$.

The discrete transitions in $\mathcal{A}'$ are defined as follows: a transition is taken from a location l_α to a location $l'_{\alpha'}$ on $(\varphi, \{n\})$ with φ defined as $n \in \alpha_i(n)$ whenever there is an ϵ transition in $\mathcal{M}_{\mathcal{A}}$ from (l, α_i) to (l', α'_j) such that (l, α_i) (respectively (l', α'_j)) is a location in the sequence l_α (respectively $l'_{\alpha'}$). The transition does not reset n if $\alpha'_j(n) \neq 0$ else if $\alpha'_j(n) = 0$ then it resets n. Formally, the definition of the one clock WIRTA is given below.

Definition 11. *Let $\mathcal{M}_{\mathcal{A}} = (Q, Q_0, \{f\}, Z, E_m, \theta_m, C_m)$ be the marked automaton corresponding to a WIRTA $\mathcal{A}$. Construct the one clock WIRTA $\mathcal{A}' = (L', L'_0, \{n\}, Z, E', \theta', C')$ as follows:*

- $L' = \{l_\alpha \mid (l, \alpha) \in Q\}$,
- $L'_0 = \{s_\alpha \mid (s, \alpha) \in Q_0\}$,
- Z= *the set of costs as in $\mathcal{M}_{\mathcal{A}}$,*
- $E' \subseteq L' \times \mathcal{G}(X') \times U_0(X') \times L'$ *consists of transitions $e = (l_\alpha, \varphi, \phi, l'_{\alpha'}) \in E'$ iff there exists $e_m = ((l, \alpha_i), \epsilon, (l', \alpha'_j)) \in E_m$ with φ defined as $n \in \alpha_i(n)$ and $\phi = \{n\}$ iff $\alpha'_j(n) = 0$, $l_\alpha[i] = (l, \alpha_i)$ and $l'_{\alpha'}[j] = (l', \alpha'_j)$*

– $\theta' : L' \to 2^{\Sigma}$ *is given as* $\theta(l_\alpha) = \theta_m(l,\alpha)$, *where* $(l,\alpha) \in Q$,
– $C' : L' \cup E' \to \mathbf{N}^{|Z|}$ *is defined as* $C'(l_\alpha) = C_m(l,\alpha)$ *where* $(l,\alpha) \in Q$, $C'(e) = C_m(e_m)$ *where* $e \in E'$ *is the edge defined above corresponding to an edge* e_m *of* E_m.

The semantics of $\mathcal{A}'$ are given by the labelled timed transition system $\mathcal{T}_{\mathcal{A}'}$ in the usual way.

A path ρ in $\mathcal{T}_{\mathcal{A}'}$ is a finite(infinite) sequence of consecutive transitions of the form $(l_\alpha,\nu_0,\mu_0) \xrightarrow{t_1} (l_\alpha,\nu_1,\mu_1) \xrightarrow{(\varphi,\phi)} (l'_{\alpha'},\nu_2,\mu_2)\dots \xrightarrow{(\varphi',\phi')} (l^n_\beta,\nu_m,\mu_m)$. Recall that $g(\rho) = (l_\alpha,t_0)(l_\alpha,t_1)(l'_{\alpha'},t_1)\dots(l^n_\beta,t_n)$. Simplifying notation, we say that $g(\rho) = (l,t_0)(l,t_1)(l',t_1)\dots(l^n,t_n)$.

Lemma 9. *Let* $\mathcal{M}_{\mathcal{A}}{=}(Q,Q_0,\{f\},Z,E_m,\theta_m,C_m)$ *be the marked automaton obtained from a WIRTA* $\mathcal{A}$ *and let* $\mathcal{A}'{=}(L',L'_0,\{n\},Z,E',\theta',C')$ *be its one clock WIRTA.*

1. *For every path* ρ_m *of* $\mathcal{T}_{\mathcal{M}}$ *where the* $\delta,\checkmark$ *strictly alternate, there exists a path* ρ *of* $\mathcal{T}_{\mathcal{A}'}$ *such that* ρ,ρ_m *are consistent.*
2. *For every path* ρ *of* $\mathcal{T}_{\mathcal{A}'}$, *there exists a path* ρ_m *of* $\mathcal{T}_{\mathcal{M}}$ *such that* ρ,ρ_m *are consistent.*
3. *Let* ρ *be a path in* $\mathcal{T}_{\mathcal{A}'}$ *consistent with a path* ρ' *in* $\mathcal{T}_{\mathcal{M}}$. *Then all paths* ρ'' *in* $\mathcal{T}_{\mathcal{A}'}$ *such that* $\rho'' \cong \rho$ *will be consistent with* ρ'.

Proof. 1. We prove by induction on the number of ϵ transitions in a path ρ of $\mathcal{T}_{\mathcal{M}}$ with strictly alternating $\delta,\checkmark$. For the base case, consider a path ρ from the initial state given by $\rho = ((l,\alpha_0),\gamma_0,\chi_0) \xrightarrow{t_{1,1}} ((l,\alpha_0),\gamma_1,\chi_1) \xrightarrow{\delta} ((l,\alpha_1),\gamma_1,\chi_1) \xrightarrow{t_{1,2}} ((l,\alpha_1),\gamma_2,\chi_2) \xrightarrow{\checkmark} ((l,\alpha_2),0,\chi_2)\dots \xrightarrow{t_{1,k+1}} ((l,\alpha_{k-1}),\gamma_k,\chi_k) \xrightarrow{\delta} ((l,\alpha_k),\gamma_k,\chi_k) \xrightarrow{\epsilon} ((l',\alpha'),\gamma',\chi')$ with k $\checkmark$'s and $k+1$ δ's. Clearly, $0 < t_{i,j} - t_{i,j-1} < 1$ for $1 \le j \le k+1$, $\alpha_0(n) = 0$, $\alpha_{k+1}(n) \in (k,k+1)$. Also, $\alpha'(n) = \alpha_{k+1}(n)$. By construction of $\mathcal{A}'$, there exists a transition $l_{\alpha_0} \xrightarrow{\varphi,\emptyset} l'_{\alpha'}$ where φ is defined as $n \in (k,k+1)$. Therefore, there is a path $\rho' = (l_{\alpha_0},\nu_0,\mu_0) \xrightarrow{t_{1,k+1}} (l_{\alpha_0},\nu'_0,\mu'_0) \xrightarrow{\varphi,\phi} (l'_{\alpha'},\nu_1,\mu_1)$ in $\mathcal{T}_{\mathcal{A}'}$ such that $\nu_0(n) \in \alpha_0(n)$, $\nu'_0(n) \in \alpha_{k+1}(n)$, and $\nu_1(n) \in \alpha'(n)$. Also, since the cost of locations l_α is same as that of (l,α), we have $\chi_0 = \mu_0$, $\chi_k = \chi_0 + C_m(l,\alpha) * t_{1,k+1} = \mu_0 + C'(l_\alpha) * t_{1,k+1} = \mu'_0$ and $\mu_1 = \mu'_0 + C(l_\alpha,\varphi,\phi,l'_{\alpha'}) = \chi_k + C_m((l,\alpha),\epsilon,(l',\alpha')) = \chi'$. Therefore, we have a path in $\mathcal{T}_{\mathcal{A}'}$ which preserves the cost.

Now, $g(\rho') = (l,0)(l,t_{1,k+1})(l',t_{1,k+1})$, $f(g(\rho')) = l\ dte(t_{1,k+1},0)\ l'$. Using Proposition 4, we have $h(\rho) = l\ dte(t_{1,k+1},0)\ l'$. Thus, $f(g(\rho')) = h(\rho)$.
The cases when the last discrete move in ρ before the ϵ happens on a $\checkmark$ can be handled similarly.
For the inductive hypothesis, assume that for all paths ρ in $\mathcal{M}_{\mathcal{A}}$ having $\le j-1$ ϵ transitions, there exist paths ρ' in $\mathcal{T}_{\mathcal{A}'}$ consistent with ρ. Consider a path ρ in $\mathcal{M}_{\mathcal{A}}$ having j ϵ transitions. We can break ρ into 2 paths ρ_1,ρ_2 having

$j-1$ ϵ transitions and one ϵ transition respectively. Using the inductive hypothesis, we obtain paths ρ_1', ρ_2' in $\mathcal{T}_{\mathcal{A}'}$ consistent with ρ_1, ρ_2. Then the path ρ' in $\mathcal{T}_{\mathcal{A}'}$ obtained by joining ρ_1' and ρ_2' will be such that $h(\rho') = f(g(\rho))$.
2. Can be proved similar to 1.
3. This is same as in Lemma 8.

□

Theorem 2. *Let $\mathcal{A}$ be a WIRTA and let $\mathcal{A}'$ be the one clock WIRTA obtained from $\mathcal{M}_{\mathcal{A}}$. Then for every path $\rho \in \mathcal{T}_{\mathcal{A}}$, there is a path ρ' in $\mathcal{T}_{\mathcal{A}'}$ such that $\rho \cong \rho'$. Further, the accumulated costs in the corresponding locations of ρ, ρ' are identical.*

Proof. The proof follows from Lemmas 8 and 9. □

Complexity

Let c_m be the highest constant that is used in the guards of a WIRTA $\mathcal{A} = (L, L_0, X, Z, E, \theta, C)$. Then for each $x \in X$, the number of intervals $\mathcal{I}_x$ is $2 * (c_m + 1)$. Thus the number of regions of $\mathcal{A}$ is $(2 * (c_m + 1))^{|X|}$. The number of locations in the marked automaton $\mathcal{M}_{\mathcal{A}} = (Q, Q_0, \{f\}, X_m, Z, E_m, \theta_m, C_m)$ is $|Q| = |L| \times (2 * c_m + 2)^{|X|+1}$. Each $\delta - \checkmark$ sequence l_α is a location in $\mathcal{A}'$. The number of $\delta - \checkmark$ sequences is $|L'| = |L| \times (2 * c_m + 2)^{|X|+1}$. Thus, starting from a WIRTA $\mathcal{A}$ with $|L|$ locations, we can obtain a one clock path preserving, cost preserving single clock WIRTA $\mathcal{A}'$ with number of locations $|L| \times (2 * c_m + 2)^{|X|+1}$.

Theorem 3. *Modelchecking $WCTL_1$ on WIRTA is decidable.*

Proof. Combine the proof of Theorem 2 with the decidability of modelchecking $WCTL_1$ on WTAs [15]. □

5 Undecidability Result

In this section, we describe two undecidability results: one for showing that $WCTL_{2r}$ model checking is undecidable on WIRTAs with 3 stopwatch costs and 1 clock, and the second for showing the undecidability of $WCTL_1$ model checking on WTAs with 2 clocks and 1 stopwatch cost.

Deterministic Two Counter Machine :

A deterministic two counter machine $\mathcal{M}$ consists of a two counters C_1 and C_2 and a finite sequence of labelled instructions. For a counter $C \in \{C_1, C_2\}$, the permitted instructions are as follows :

1. l_i : *goto* l_k
2. l_i : $C = C + 1$
3. l_i : $C = C - 1$
4. l_i : *if* $C = 0$ *goto* l_i^1 *else goto* l_i^2
5. l_i : *halt*

Without loss of generality, assume that the instructions are labelled $l_1, \ldots, l_n$ where $l_n = HALT$ (a special instruction) and that in the initial configuration, both counters have value zero. The behavior of the machine is described by a possibly infinite sequence of configurations $\langle l_1, 0, 0\rangle, \langle l_1, C_1^1, C_2^1\rangle, \ldots \langle l_k, C_1^k, C_2^k\rangle \ldots$ where C_1^k and C_2^k are the respective counter values and l_k is the label of the kth instruction. The halting problem of such a machine is known to be undecidable [24].

In the following sections, we show that model checking of some of the subclasses of the logics introduced in the Section 2.3 over WTA is undeciable. Some of the results are over W-IRTA and due to the result of Section 3.1, only a single clock has been considered. Note that in the following sections we shall represent the state (l, ν, μ) of a WTA as $(l, \langle \nu(x_1), \nu(x_2) \cdots \nu(x_n)\rangle, \langle \mu(z_1), \mu(z_2) \cdots \mu(z_m)\rangle)$. To indicate that the value of a component is not important, a special symbol $-$ would be used. Additionally, transitions representing elpase of time t in a location are shown as $(l, \nu, \mu) \xrightarrow{t} (l, \nu', \mu')$.

Lemma 10. *Model checking $WCTL_2$ on WIRTA with 1 clock and 3 stopwatch costs is undecidable.*

The proof given in [17] holds for a WRITA with minimal modifications. The constraint $x = 1?$ is replaced by $x = 1?x := 0$ while $x = 0?$ is the constraint over all the other edges.

Model checking $WCTL_{2r}$ on WIRTAs with 1 clock and 3 stopwatch costs

A WIRTA $\mathcal{A}$=$(L, \{l_1\}, X, Z, E, \theta, C)$ and a $WCTL_{2r}$ formula Ψ are used to simulate a deterministic counter machine M. $X = \{x\}$, $Z = \{z_1, z_2, z_3\}$ where

$z_i, 1 \leq i \leq 3$ is a stopwatch and $\theta(l_i) = l_i$. The normal form of the vairables is $x = 0$, $z_3 = 0$, $z_1 = 1 - \frac{1}{2^{n_1} * 3^{n_2}}$ and $z_2 = 1 - \frac{1}{2^{n_3} * 3^{n_4}}$ where $1 \leq i \leq 4, n_i \geq 0$ encode the counters of M as $C_1 = n_1 - n_2$ and $C_2 = n_3 - n_4$. Each instruction l_i of M is simulated by a sub-automaton $\mathcal{A}_i$ and a $WCTL_{2r}$ formula. $\mathcal{A}_i$ (which is a WIRTA) is built such that the initial location of $\mathcal{A}_i$ is labelled l_i. For the last instruction $l_n :: HALT$, the sub-automata has a single state with the label $HALT$. The values of all the variables are in their respective normal forms in each l_i. Also the stopwatch z_3 and clock constraint $x = 0?$ ensure that no time elapses in certain locations. Now, the final formula is given by $\Psi :: z_1.z_2.z_3.\mathbf{E}\ \psi_{all}\ \mathbf{U}\ (HALT\ \wedge\ z_3 = 0)$, ψ_{all} will be given at the end of this section. The final WIRTA $\mathcal{A}$ is obtained by connecting all the sub-automata $\mathcal{A}_i$ such that the locations l_i in the automata $\mathcal{A}_{i-1}$ and $\mathcal{A}_i$ coincide. The initial values are $C_1 = 0, C_2 = 0$ in M. Thus $n_1 = n_2 = 0$ and $n_3 = n_4 = 0$. Hence all the variables are in their normal form in the start location l_1. The module for instruction $l_i :: goto\ l_k$ is given in Figure 5.0.1.

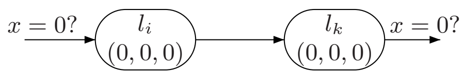


Fig. 5.0.1. $WCTL_{2r}$ and WIRTA with 1 clock and 3 stopwatches : Module for instruction $l_i ::\ goto\ l_k$.

The instructions of M are simulated as follows.

1. Increment C_1 : Increment n_1 by adding $\frac{1}{2^{n_1+1} * 3^{n_2}}$ to $z_1 = 1 - \frac{1}{2^{n_1} * 3^{n_2}}$.
2. Decrement C_2 : Increment n_2 by adding $\frac{2}{3} * \frac{1}{2^{n_1} * 3^{n_2}}$ to $z_1 = 1 - \frac{1}{2^{n_1} * 3^{n_2}}$.
3. Checking if C_1 is zero : $C_1 = 0$ iff $n_1 = n_2$. This is achieved by multipying the value $\frac{1}{2^{n_1} * 3^{n_2}}$ by 6 an integral number of times till it becomes 1. Multiplying $\frac{1}{2^{n_1} * 3^{n_2}}$ by 6 once decrements both n_1 and n_2.
4. Identical operations can be performed with respect to counter C_2 by reversing the roles of z_1 and z_2 in all the modules pertaining to C_1.

The module to increment C_1 is given in Figure 5.0.2. The formula $\psi_{I_1} :: I_1 \Longrightarrow \mathbf{E}\ I_1\ \mathbf{U}\ (\neg I_1\ \wedge\ z_3 = 0\ \wedge\ \psi_{check\_z\_2c})$ ensures that the amount of time spent in the location q_1 is $\frac{1}{2} * (1 - z_1)$.

Lemma 11. *In the module $\mathcal{A}_i$ given in the Figure 5.0.2, there exists a path from $(l_i, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}}, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle)$ to $(l_{i+1}, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle)$ witnessing ψ_{I_1} iff $t = \frac{1}{2^{n_1+1} * 3^{n_2}}$. (Due to the final formula Ψ, $z_3 = 0$ in l_{i+1} .)*

Proof. Consider a path ρ_{inc} from $(l_i, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}}, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle)$ to $(l_{i+1}, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle)$. $\rho_{inc} = (l_i, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}}, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle) \xrightarrow{x=0?} (q_1, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}}, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle) \xrightarrow{t} (q_1, t, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle) \longrightarrow (I_1, t, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle) \longrightarrow (q_2, t, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle) \xrightarrow{1-t} (q_2, 1, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 - \frac{1}{2^{n_3} * 3^{n_4}}, 0\rangle) \xrightarrow{x=1?x:=0} (l_{i+1}, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, 1 -$

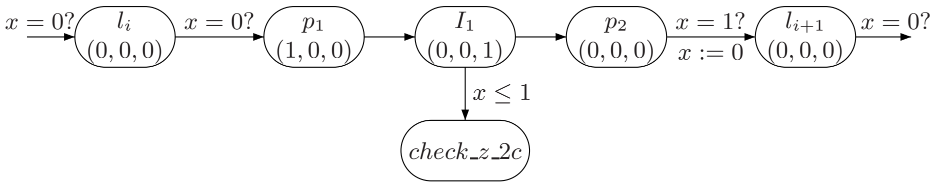


Fig. 5.0.2. $WCTL_{2r}$ and WIRTA with 1 clock and 3 stopwatches : Module $\mathcal{A}_i$ and $\psi_{I_1} :: I_1 \Longrightarrow \mathbf{E}\, I_1 \mathbf{U}(\neg\, I_1 \,\wedge\, z_3 = 0 \,\wedge\, \psi_{check\_z\_2c})$ to increment C_1. Module *check_z_2c* is given in Figure 5.0.3.

$\frac{1}{2^{n_3}*3^{n_4}}, 0\rangle)$.
It is clear that ρ_{inc} is a witness of ψ_{I_1} iff $(I_1, t, \langle 1 - \frac{1}{2^{n_1}*3^{n_2}} + t, 1 - \frac{1}{2^{n_3}*3^{n_4}}, 0\rangle) \models \psi_{I_1}$. From Lemma 15, it follows that ψ_{I_1} holds at $(I_1, t, \langle 1 - \frac{1}{2^{n_1}*3^{n_2}} + t, 1 - \frac{1}{2^{n_3}*3^{n_4}}, 0\rangle)$ iff $t = \frac{1}{2^{n_1+1}*3^{n_2}}$. □

Lemma 12. *In the module get_x^2 in Figure 5.0.3, there exists a path from $(q_1, t, \langle -, 0, 0\rangle)$ to $(q_2, t', \langle -, 1, t'\rangle)$ iff $t' = t$.*

Follows from simple calculations.

Lemma 13. *In the module Add in Figure 5.0.3, there exists a path from $(a_1, -, \langle \delta_1, \delta_2, 0\rangle)$ to $(A, t_a, \langle \delta_1, \delta_2 + t_a, 0\rangle)$ witnessing ψ_A iff $t_a = \delta_1$.*

Proof. Consider a path ρ_A from $(a_1, -, \langle \delta_1, \delta_2, 0\rangle)$ to $(A, t_a, \langle \delta_1, \delta_2 + t_a, 0\rangle)$. The value of x is the time t_a spent in a_2 iff no time elpases in A. It is clear from the module that $(A, t_a, \langle \delta_1, \delta_2 + t_a, 0\rangle) \models \psi_A$ iff $t_a = \delta_1$ as upon reaching A_F, $z_1 = \delta_1 + t' = 1$ and $x = t_a + t' = 1$. Here t' is the time spent in location a_3. □

Lemma 14. *In the module check in Figure 5.0.3, if the initial values were $z_1 = 1 - \alpha + t$, $z_2 = t_3$ and $z_3 = t$ then ψ_{C_2} holds iff $t_3 = \alpha$.*

Proof. Consider a path ρ_{C_2} from $(C_1, \zeta, \langle 1 - \alpha + t, t_3, t\rangle)$ to location C_{2F}. $\psi_{C_2} = (C_1, \zeta, \langle 1 - \alpha + t, t_3, t\rangle) \xrightarrow{2-\zeta} (C_1, 2, \langle 1 - \alpha + t, t_3, t\rangle) \xrightarrow{x=2?x:=0} (C_2, 0, \langle 1 - \alpha + t, t_3, t\rangle) \xrightarrow{t_{22}} (C_2, t_{22}, \langle 1 - \alpha + t + t_{22}, t_3, t\rangle) \longrightarrow (C_3, t_{22}, \langle 1-\alpha+t+t_{22}, t_3, t\rangle) \xrightarrow{1-t_{22}} (C_3, 1, \langle 1-\alpha+t+t_{22}, t_3+1-t_{22}, t\rangle) \xrightarrow{x=1?x:=0} (C_4, 0, \langle 1-\alpha+t+t_{22}, t_3+1-t_{22}, t\rangle) \xrightarrow{1} (C_4, 1, \langle 1-\alpha+t+t_{22}, t_3+1-t_{22}, t+1\rangle) \longrightarrow (C_5, 1, \langle 1-\alpha+t+t_{22}, t_3+1-t_{22}, t+1\rangle) \xrightarrow{t_{25}} (C_5, 1+t_{25}, \langle 1-\alpha+t+t_{22}+t_{25}, t_3+1-t_{22}, t+1+t_{25}\rangle) \longrightarrow (C_{2F}, 1+t_{25}, \langle 1-\alpha+t+t_{22}+t_{25}, t_3+1-t_{22}, t+1+t_{25}\rangle)$.

ρ_{C_2} is a witness of ψ_{C_2} iff $z_1 = 1 - \alpha + t + t_{22} + t_{25} = 2$ and $z_2 = t_3 + 1 - t_{22} = 1$ and $z_3 = t + 1 + t_{25} = 2$. Thus $t_{22} = t_3$ and $t_{25} = 1 - t$ and $1 - \alpha + t + t_3 + 1 - t = 2 \iff t_3 = \alpha$. It follows from simple calculations that ρ_{C_2} is a witness of ψ_{C_2} iff $t_3 = \alpha$.

Thus this module checks if the value in z_2 is $1 - z_1 + z_3$. □

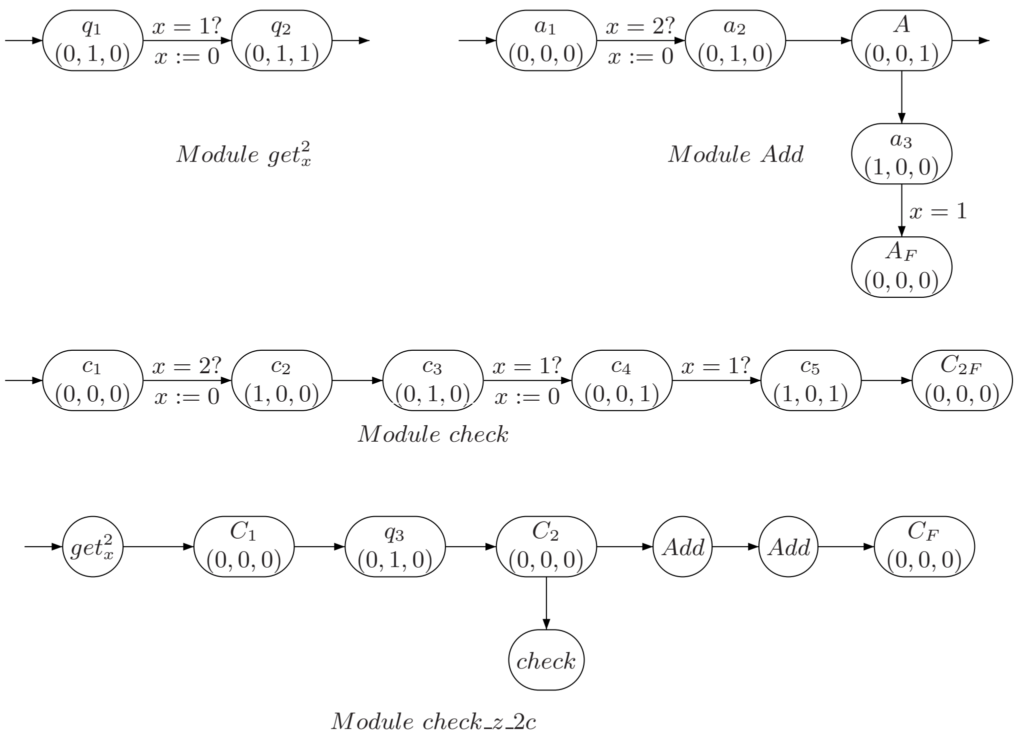


Fig. 5.0.3. WC_TL_{2r} and WIRTA with 1 clock and 3 stopwatches : Module *Add* and formula $\psi_A :: (A \wedge z_3 = 0) \Longrightarrow \mathbf{E}\ \neg A_F\ \mathbf{U}\ (A_F \wedge z_1 = 1 \wedge z_3 = 0)$ ensure that time spent in location a_2 is the same as the value in z_1. Module *check* is associated with the formula $\psi_{C_2} :: C_2 \Longrightarrow \mathbf{E}\ \neg C_{2F}\ \mathbf{U}\ (C_{2F} \wedge z_2 = 1 \wedge z_1 = 2 \wedge z_3 = 2)$. Module *check_z_2c* is associated with the formula $\psi_{check\_z\_2c} :: z_2.z_3.\ \mathbf{E}\ \neg C_1\ \mathbf{U}\ [C_1 \wedge z_2 = 1 \wedge z_2.\ \mathbf{E}\ \neg C_2\ \mathbf{U}\ \{C_2 \wedge \psi_{C_2} \wedge z_3.\ \mathbf{E}\ (\neg C_F \wedge \psi_A)\ \mathbf{U}\ (C_F \wedge z_2 = 2 \wedge z_3 = 0)\}]$

Lemma 15. *In the module check_z_2c in Figure 5.0.3, if the initial values were* $z_1 = 1-\alpha+t$ *and* $x = t$ *then* $\psi_{check\_z\_2c}$ *holds iff* $t = \frac{\alpha}{2}$. *(Note that* $0 \leq \alpha, t \leq 1$*)*

Proof. In the module *check_z_2c*, along a path from location q_1 to C_F, the value in $z_2 = \alpha$ before entering the first *Add* module (from Lemmas 12 and 14). Due to Lemma 13, the value in $z_2 = \alpha+1-\alpha+t+1-\alpha+t$ when C_F is reached. Such a path is a witness of $\psi_{check\_z\_2c}$ iff $z_2 = \alpha+1-\alpha+t+1-\alpha+t = 2 \iff t = \frac{\alpha}{2}$. □

Proof. Consider a path $\rho_{check\_z\_2c}$ from $(q_1, t, \langle 1-\alpha+t, 0, 0\rangle)$ to location C_F. $\rho_{check\_z\_2c} = (q_1, t, \langle 1-\alpha+t, 0, 0\rangle) \xrightarrow{1-t} (q_1, 1, \langle 1-\alpha+t, 1-t, 0\rangle) \xrightarrow{x=1?x:=0} (q_2, 0, \langle 1-\alpha+t, 1-t, 0\rangle) \xrightarrow{t'} (q_2, t', \langle 1-\alpha+t, 1-t+t', t'\rangle) \longrightarrow (C_1, t', \langle 1-\alpha+t, 1-t+t', t'\rangle)\cdots$. $\rho_{check\_z\_2c}$ is a witness of $\psi_{check\_z\_2c}$ iff $1-t+t' = 1$ and $(C_1, t', \langle 1-\alpha+t, 0, t'\rangle) \models \mathbf{E}\, \neg C_2\, \mathbf{U}\, \{C_2 \wedge \psi_{C_2} \wedge z_3.\, \mathbf{E}\, (\neg C_F \wedge \psi_A)\, \mathbf{U}\, (C_F \wedge z_2 = 2 \wedge z_3 = 0)\}$. From Lemma 12, it follows that $t' = t$.

Consider a path ρ_{C_1} starting from $(C_1, t, \langle 1-\alpha+t, 0, t\rangle)$. As it is not going to affect the stopwatch costs, let the time spent in locations C_1 and C_2 be 0. $\rho_{C_1} = (C_1, t, \langle 1-\alpha+t, 0, t\rangle) \longrightarrow (q_3, t, \langle 1-\alpha+t, 0, t\rangle) \xrightarrow{t_3} (q_3, t+t_3, \langle 1-\alpha+t, t_3, t\rangle) \longrightarrow (C_2, t+t_3, \langle 1-\alpha+t, t_3, t\rangle)$. ρ_{C_1} is a witness of $\mathbf{E}\, \neg C_2\, \mathbf{U}\, \{C_2 \wedge \psi_{C_2} \wedge z_3.\, \mathbf{E}\, (\neg C_F \wedge \psi_A)\, \mathbf{U}\, (C_F \wedge z_2 = 2 \wedge z_3 = 0)\}$ iff $(C_2, t+t_3, \langle 1-\alpha+t, t_3, t\rangle) \models \psi_{C_2}$ and $(C_2, t+t_3, \langle 1-\alpha+t, t_3, 0\rangle) \models \mathbf{E}\, (\neg C_F \wedge \psi_A)\, \mathbf{U}\, (C_F \wedge z_2 = 2 \wedge z_3 = 0)$. From Lemma 14, it is clear that $t_3 = \alpha$.

Consider a path ρ_{C_F} from $(C_2, t+\alpha, \langle 1-\alpha+t, \alpha, 0\rangle)$ to location C_F. $\rho_{C_F} = (C_2, t+\alpha, \langle 1-\alpha+t, \alpha, 0\rangle) \longrightarrow (a_1, t+\alpha, \langle 1-\alpha+t, \alpha, 0\rangle)\rho_A(A, 1-\alpha+t, \langle 1-\alpha+t, \alpha+1-\alpha+t, 0\rangle)\rho_A(A, 1-\alpha+t, \langle 1-\alpha+t, \alpha+1-\alpha+t+1-\alpha+t, 0\rangle) \longrightarrow (C_F, 1-\alpha+t, \langle 1-\alpha+t, \alpha+1-\alpha+t+1-\alpha+t, 0\rangle)$. ρ_A is the path as in Lemma 13.

ρ_{C_F} is a witness of $\mathbf{E}\, (\neg C_F \wedge \psi_A)\, \mathbf{U}\, (C_F \wedge z_2 = 2 \wedge z_3 = 0)$ iff $z_2 = \alpha+1-\alpha+t+1-\alpha+t = 2 \iff t = \frac{\alpha}{2}$. (Note that no time elapse has been considered in locations A as $z_3 = 0$ upon reaching C_F.)

Thus for the module *check_z_2c* if the initial values were $z_1 = 1-\alpha+t$ and $x = t$ then $\psi_{check\_z\_2c}$ holds iff $t = \frac{\alpha}{2}$. □

In module *check_z_2c* of Figure 5.0.3, it is important to ensure that the value of x does not go beyond 1 (2) before the reset $x = 1?x := 0$ ($x = 2?x := 0$) even when it is not accounted for in the calculations. From Figure 5.0.2, it is clear that when the module *check_z_2c* is entered, $0 \leq x \leq 1$. Thus the total time elpased in locations q_2 and q_3 is atmost 2. Similar argument holds in all the resets in other modules too.

Module to increment n_2 is similar to that in Figure 5.0.2 with the I_1 replaced by D_1, *check_z_2c* replaced by another module *check_z_3c*. This module is obtained from *check_z_2c* by an additional *Add* before location C_F. The formula $\psi_{check\_z\_3c}$ would compare $z_2 = 3$ instead of $z_2 = 2$ as in $\psi_{check\_z\_2c}$. As a consequence of this modification $z_2 = \alpha + 1 - \alpha + t + 1 - \alpha + t + 1 - \alpha + t = 3$ when C_F is reached, which in turn ensures that $t = \frac{2}{3} * \alpha$. Thus the formula to decrement counter C_1 is $\psi_{D_1} :: D_1 \Longrightarrow \mathbf{E}\, D_1\, \mathbf{U}\, (\neg D_1 \wedge z_3 = 0 \wedge \psi_{check\_z\_3c})$.

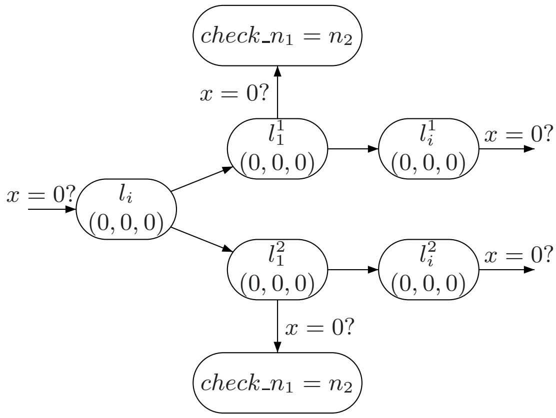


Fig. 5.0.4. *$WCTL_{2r}$* and WIRTA with 1 clock and 3 stopwatches : Module $\mathcal{A}_z$ and formula ψ_{Z_1} check if counter C_1 is zero. $\psi_{Z_1} :: (l_1^1 \Longrightarrow \psi_{check\_n_1=n_2}) \wedge (l_1^2 \Longrightarrow \neg\psi_{check\_n_1=n_2})$. Module $check\_n_1 = n_2$ is given in Figure 5.0.5

The module $\mathcal{A}_z$ given in Figure 5.0.4 and the formula ψ_{Z_1} together ensure that C_1 is zero iff location q_i^1 is reached (else q_i^2 is reached). From the encoding of C_1 as $n_1 - n_2$ it follows that $C_1 = 0$ iff $n_1 = n_2$. One way of checking whether $n_1 = n_2$ is to simultaneousy decrement both n_1 and n_2 and check if both of them are zero after each step. As both n_1 and n_2 are encoded by the stopwatch $z_1 = 1 - \frac{1}{2^{n_1} * 3^{n_2}}$, simultaneous decrement of n_1 and n_2 is achieved by first extracting $\frac{1}{2^{n_1} * 3^{n_2}}$ into stopwatch z_2 and adding to z_2 its own value to decrement n_1 and then adding $2 * z_2$ to decrement n_2. Thus decrementing n_1 and n_2 once, leaves z_2 with $6 * \frac{1}{2^{n_1} * 3^{n_2}} = \frac{1}{2^{n_1-1} * 3^{n_2-1}}$. This is done in a loop untill $z_2 = 1$. After i loops, if $z_2 = 1$ then $6^i * \frac{1}{2^{n_1} * 3^{n_2}} = 1$, thus indicating that $n_1 = n_2$. The module $check\_n_1 = n_2 = i$ in Figure 5.0.5 does the above calculation.

Lemma 16. *In the module $check\_z_2 = 1 - z_1$ in Figure 5.0.5, if the initial values were $z_1 = 1 - \alpha$, $z_2 = t$ and $z_3 = 0$ then ψ_{M_1} holds iff $t = \alpha$.*

Proof. Consider a path $\rho_{M_1} = (m_1, -, \langle 1 - \alpha, t, 0\rangle) \xrightarrow{t_{11}} (m_1, -, \langle 1 - \alpha, t + t_{11}, t_{11}\rangle) \longrightarrow (m_2, -, \langle 1 - \alpha, t + t_{11}, t_{11}\rangle) \xrightarrow{t_{12}} (m_2, -, \langle 1 - \alpha + t_{12}, t + t_{11}, t_{11} + t_{12}\rangle) \longrightarrow (M_{1F}, -, \langle 1 - \alpha + t_{12}, t + t_{11}, t_{11} + t_{12}\rangle)$.

ρ_{M_1} is a witness of ψ_{M_1} iff $z_1 = 1 - \alpha + t_{12} = 1$ and $z_2 = t + t_{11} = 1$ and $z_3 = t_{11} + t_{12} = 1$. Thus $t = \alpha$. □

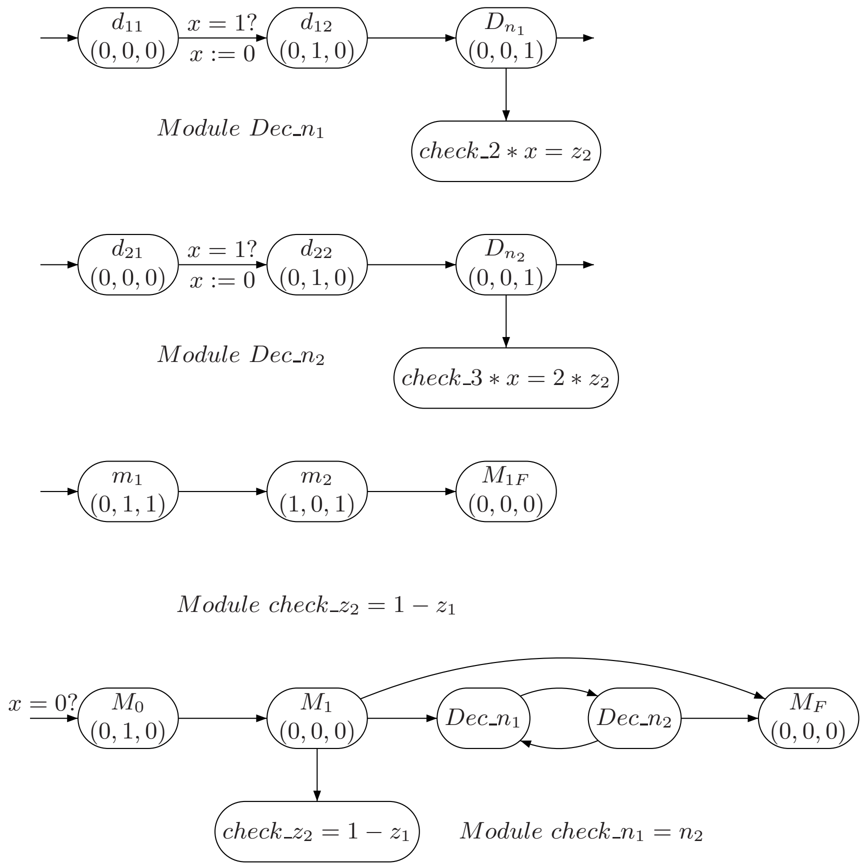


Fig. 5.0.5. $WCTL_{2r}$ and WIRTA with 1 clock and 3 stopwatches : Module $check\_n_1 = n_2$ checks if $n_1 = n_2$ with the help of the formula $\psi_{check\_n_1=n_2}$ :: $z_2.z_3$. $\mathbf{E}\ (\neg M_F \ \wedge\ \psi_{M_1} \ \wedge\ \psi_{D_{n_1}} \ \wedge\ \psi_{D_{n_2}})\mathbf{U}(M_F \ \wedge\ z_3 = 0 \ \wedge\ z_2 = 1)$. Here ψ_{M_1} :: $M_1 \Longrightarrow \mathbf{E}\ \neg M_{1F}\mathbf{U}(M_{1F} \ \wedge\ z_2 = 1 \ \wedge\ z_3 = 1 \ \wedge\ z_1 = 1)$ pertains to the module $check\_z_2 = 1 - z_1$. The formulae $\psi_{D_{n_1}}$ :: $D_{n_1} \Longrightarrow \mathbf{E}\ D_{n_2}\mathbf{U}(\neg D_{n_2} \ \wedge\ z_3 = 0 \wedge \psi_{check\_2*x=z_2})$ and $\psi_{D_{n_2}}$ :: $D_{n_2} \Longrightarrow \mathbf{E}\ D_{n_1}\mathbf{U}(\neg D_{n_1} \ \wedge\ z_3 = 0 \ \wedge\ \psi_{check\_3*x=2*z_2})$ ensure that n_1 and n_2 are decremented respectively. Module $check\_3 * x = 2 * z_2$ is given in Figure 5.0.7 and Module $check\_2 * x = z_2$ is in Figure 5.0.6.

Lemma 17. *In the module Dec_n_1 in Figure 5.0.5, there exists a path from $(d_{11}, -, \langle -, \alpha, 0\rangle)$ to $(D_{n_1}, t_{d1}, \langle -, \alpha + t_{d1}, 0\rangle)$ witnessing $\psi_{D_{n_1}}$ iff $t_{d1} = \alpha$.*
In the module Dec_n_2 in Figure 5.0.5, there exists a path from $(d_{21}, -, \langle -, \beta, 0\rangle)$ to $(D_{n_2}, t_{d1}, \langle -, \beta + t_{d2}, 0\rangle)$ witnessing $\psi_{D_{n_2}}$ iff $t_{d2} = 2 * \beta$.

Proof. As no time elpases in locations D_{n_1} and D_{n_2} (due to $\psi_{D_{n_1}}$ and $\psi_{D_{n_2}}$), $t_{d1} = \alpha$ and $t_{d2} = 2 * \beta$ follow from Lemmas 20 and 23. □

Lemma 18. *In the module $check\_n_1 = n_2$ in Figure 5.0.5, if the initial values were $z_1 = 1 - \frac{1}{2^{n_1} * 3^{n_2}}$ then $\psi_{check\_n_1=n_2}$ holds iff $n_1 = n_2$ (that is iff $C_1 = 0$).*

Proof. For simplicity of argument assume that no time elpases in locations D_{n_1} and D_{n_2}. Note that this only simplifies the paths we consider without affecting the correctness of the argument. ($z_3 = 0$ clause in $\psi_{check\_n_1=n_2}$ upon reaching M_F ensures that no time elpases in these locations anyways.) Also as it does not affect our computations in assume that no time elpases in location M_1 too.

Let $\alpha = \frac{1}{2^{n_1} * 3^{n_2}}$. Consider a path $\rho_{check\_n_1=n_2}$ from $(M_0, 0, \langle 1 - \alpha, 0, 0\rangle)$ to location M_F. $\rho_{check\_n_1=n_2} = (M_0, 0, \langle 1 - \alpha, 0, 0\rangle) \xrightarrow{t} (M_0, t, \langle 1 - \alpha, t, 0\rangle) \longrightarrow (M_1, t, \langle 1 - \alpha, t, 0\rangle)\rho_{end}$.
$\rho_{check\_n_1=n_2}$ is a witness of $\psi_{check\_n_1=n_2}$ iff

1. $(M_1, t, \langle 1 - \alpha, t, 0\rangle) \models \psi_{M_1} \iff t = \alpha$ from Lemma 16.
2. For every occurence of locations D_{n_1} and D_{n_2}, $(D_{n_1}, t_{d1}, \langle 1 - \alpha, t + t_{d1}, 0\rangle) \models \psi_{D_{n_1}}$ and $(D_{n_2}, t_{d2}, \langle 1 - \alpha, t + t_{d1} + t_{d2}, 0\rangle) \models \psi_{D_{n_2}} \iff t + t_{d1} = 2 * \alpha$ and $t + t_{d1} + t_{d2} = 2 * \alpha + 2 * (2 * \alpha) = 6 * \alpha$ from Lemma 17.
3. $\rho_{end} = (M_F, t, \langle 1 - \alpha, t, 0\rangle)$ (if $n_1 = n_2 = 0$ then $t = \alpha = 1$) or $\rho_{end} = (d_{11}, t, \langle 1 - \alpha, t, 0\rangle) \overset{\rho_{d1}}{\dashrightarrow} (D_{n_1}, t_{d1}, \langle 1 - \alpha, t + t_{d1}, 0\rangle) \longrightarrow (d_{21}, t_{d1}, \langle 1 - \alpha, t + t_{d1}, 0\rangle) \overset{\rho_{d2}}{\dashrightarrow} (D_{n_2}, t_{d2}, \langle 1 - \alpha, t + t_{d1} + t_{d2}, 0\rangle)\rho_{end}$. (Here ρ_{d1} and ρ_{d2} are paths in modules Dec_n_1 and Dec_n_2 respectively.)
   Thus after looping i times if $(M_F, -, \langle 1 - \alpha, 1, 0\rangle)$ is reached then $6^i * \alpha = 1 \iff n_1 = n_2$.

□

Lemma 19. *In the module $check\_x = z_3$ in Figure 5.0.6, if the initial values were $z_3 = t$, $z_1 = 0$ and $x = t_2$ then $\psi_{check\_x=z_3}$ holds iff $t_2 = \alpha$.*

Follows from simple calculations.

Lemma 20. *In the module $check\_2 * x = z_2$ in Figure 5.0.6, if the initial values were $z_2 = \alpha + t$ and $x = t$ then $\psi_{check\_2*x=z_2}$ holds iff $t = \alpha$.*

Proof. Consider a path $\rho_{check\_2*x=z_2}$ from $(q_1, t, \langle 0, \alpha + t, 0\rangle)$ to location B_F. $\rho_{check\_2*x=z_2} = (q_1, t, \langle 0, \alpha + t, 0\rangle) \overset{\rho_{get^1_x}}{\dashrightarrow} (B_1, t', \langle 1 - t + t', \alpha + t, t'\rangle)$. Then $\rho_{check\_2*x=z_2}$ is a witness of $\psi_{check\_2*x=z_2}$ iff $z_1 = 1 - t + t' = 1$ and

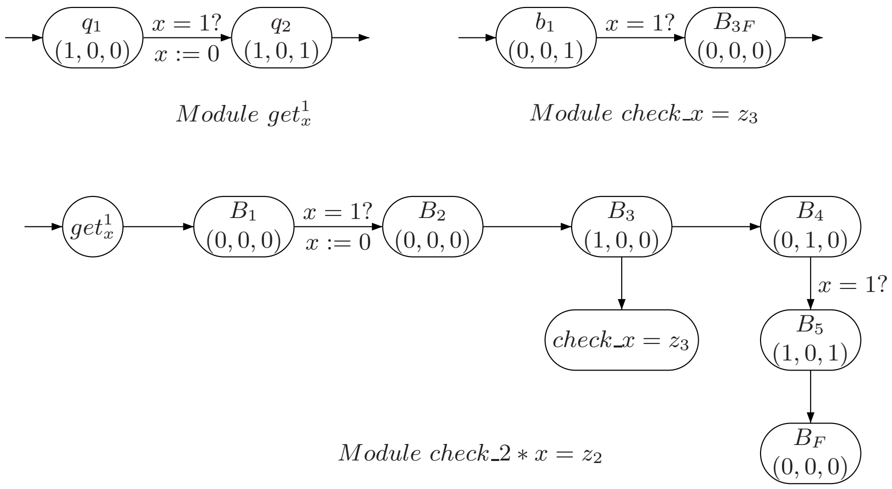


Fig. 5.0.6. $WCTL_{2r}$ and WIRTA with 1 clock and 3 stopwatches : Module *check_2* $*$ $x = z_2$ and formula $\psi_{check\_2*x=z_2}$ check if $t = \alpha$ given that $x = t$ and $z_2 = \alpha + t$ while entering the module. Here $\psi_{check\_2*x=z_2}$ :: $z_1.z_3$. $\mathbf{E}\ \neg B_1\ \mathbf{U}\ [B \ \wedge\ z_1 = 1 \ \wedge\ z_1.\ \mathbf{E}(\neg B_F \ \wedge\ \psi_{check\_x=z_3})\ \mathbf{U}\ (B_F \ \wedge\ z_1 = 1 \ \wedge\ z_2 = 1 \ \wedge\ z_3 = 1)]$ and $\psi_{check\_x=z_3} :: (B_3 \ \wedge\ z_1 = 0) \Longrightarrow \mathbf{E}\ \neg B_{3F}\ \mathbf{U}\ (B_{3F} \ \wedge\ z_3 = 1 \ \wedge\ z_1 = 0)$.

$(B_1, t', \langle 0, \alpha{+}t, t'\rangle) \models \mathbf{E}(\neg B_F \wedge \psi_{check\_x=z_3})\ \mathbf{U}\ (B_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)$. From argument similar to Lemma 12, it is clear that $t' = t$.

Consider a path ρ_{B_F} from $(B_1, t, \langle 0, \alpha + t, t\rangle)$ to location B_F. $\rho_{B_F} = (B_1, t, \langle 0, \alpha{+}t, t\rangle) \xrightarrow{1-t} (B_1, 1, \langle 0, \alpha{+}t, t\rangle) \xrightarrow{x=1?x:=0} (B_2, 0, \langle 0, \alpha{+}t, t\rangle) \xrightarrow{t_{12}} (B_2, t_{12}, \langle 0, \alpha{+}t, t\rangle) \longrightarrow (B_3, t_{12}, \langle 0, \alpha{+}t, t\rangle) \xrightarrow{t_{13}} (B_3, t_{12}{+}t_{13}, \langle t_{13}, \alpha{+}t, t\rangle) \longrightarrow (B_4, t_{12} + t_{13}, \langle t_{13}, \alpha + t, t\rangle) \xrightarrow{t_{14}} (B_4, t_{12} + t_{13} + t_{14}, \langle t_{13}, \alpha + t + t_{14}, t\rangle) \xrightarrow{x=1?} (B_5, 1, \langle t_{13}, \alpha + t + t_{14}, t\rangle) \xrightarrow{t_{15}} (B_5, 1 + t_{15}, \langle t_{13} + t_{15}, \alpha + t + t_{14}, t + t_{15}\rangle) \longrightarrow (B_F, 1 + t_{15}, \langle t_{13} + t_{15}, \alpha + t + t_{14}, t + t_{15}\rangle)$.

ρ_{B_F} is a witness of $\mathbf{E}(\neg B_F \wedge \psi_{check\_x=z_3})\ \mathbf{U}\ (B_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)$ iff

1. $(B_3, t_{12}, \langle 0, \alpha + t, t\rangle) \models \psi_{check\_x=z_3}$ and
2. $t_{12} + t_{13} + t_{14} = 1$ ($x = 1?$ transition from B_4 to B_5) $z_1 = t_{13} + t_{15} = 1$ and $z_2 = \alpha + t + t_{14} = 1$ and $z_3 = t + t_{15} = 1$.

From Lemma 19 and the above equations it is clear that ρ_{B_F} is a witness iff $t = \alpha$. $\square$

Lemma 21. *In the module* $check\_z_1 = z_2 - z_3$ *in Figure 5.0.7, if the initial values were* $z_2 = \beta + t$, $z_3 = t$ *and* $z_1 = t_2$ *then* $\psi_{check\_z_1=z_2-z_3}$ *holds iff* $t_2 = \beta$.

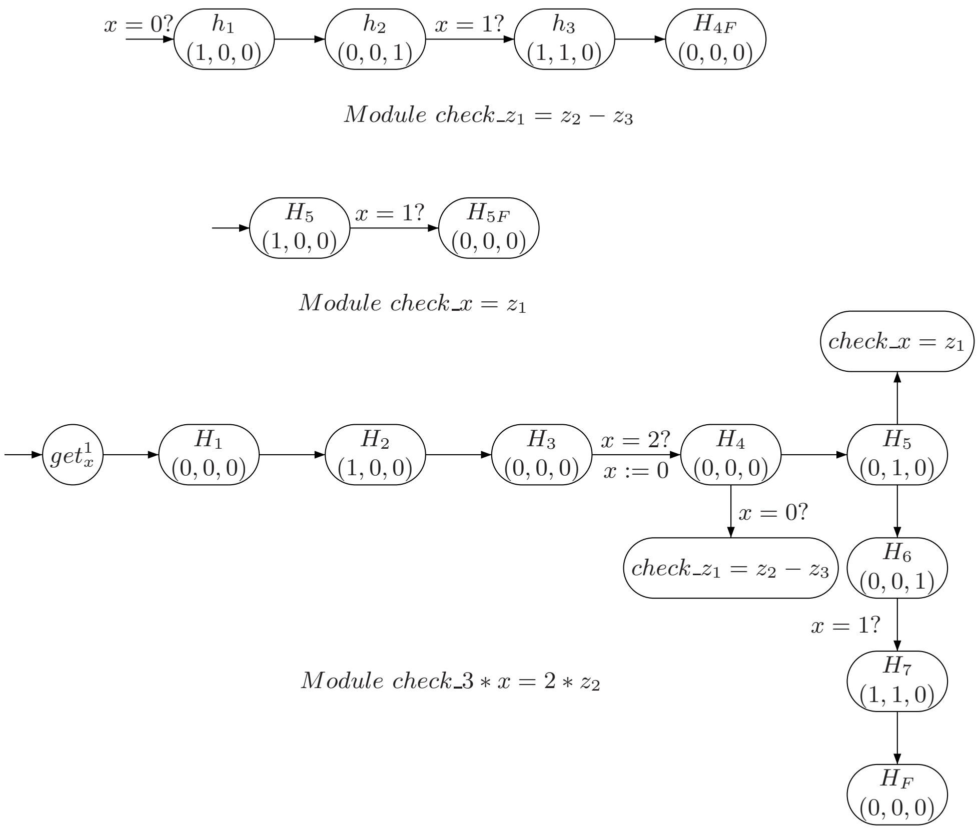


Fig. 5.0.7. $WCTL_{2r}$ and WIRTA with 1 clock and 3 stopwatches : Module $check\_3 * x = 2 * z_2$ and formula $\psi_{check\_3*x=2*z_2}$ ensure that $t = 2 * \beta$ given that the initial values were $x = t$ and $z_2 = \beta + t$. The formula is $\psi_{check\_3*x=2*z_2} :: z_1.z_3.\ \mathbf{E}\ \neg H_1\ \mathbf{U}\ [H_1 \wedge z_1 = 1 \wedge z_1.\ \mathbf{E}\ \neg H_4\ \mathbf{U}\ \{H_4 \wedge \psi_{check\_z_1=z_2-z_3} \wedge z_2.\ \mathbf{E}(\neg H_F \wedge \psi_{check\_x=z_1})\ \mathbf{U}\ (H_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)\}]$. Here $\psi_{check\_z_1=z_2-z_3} :: H_4 \Longrightarrow \mathbf{E}\ \neg H_{4F}\ \mathbf{U}\ (H_{4F} \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)$ and $\psi_{check\_x=z_1} :: (H_5 \wedge z_2 = 0) \Longrightarrow \mathbf{E}\ H_5\ \mathbf{U}\ (H_{5F} \wedge z_1 = 1 \wedge z_2 = 0)$. Module get_x^1 is in Figure 5.0.6.

Proof. Let $\rho_{check\_z_1=z_2-z_3}$ be the path from $(h_1, 0, \langle t_{22}, \beta + t, t\rangle)$ to location H_{4F}. $\rho_{check\_z_1=z_2-z_3} = (h_1, 0, \langle t_{22}, \beta + t, t\rangle) \xrightarrow{t'_1} (h_1, t'_1, \langle t_{22} + t'_1, \beta + t, t\rangle) \longrightarrow (h_2, t'_1, \langle t_{22} + t'_1, \beta + t, t\rangle) \xrightarrow{t'_2} (h_2, t'_1 + t'_2, \langle t_{22} + t'_1, \beta + t, t + t'_2\rangle) \xrightarrow{x=1?} (h_3, 1, \langle t_{22} + t'_1, \beta + t, t + t'_2\rangle) \xrightarrow{t'_3} (h_3, 1 + t'_3, \langle t_{22} + t'_1 + t'_3, \beta + t + t'_3, t + t'_2\rangle) \longrightarrow (H_{4F}, 1 + t'_3, \langle t_{22} + t'_1 + t'_3, \beta + t + t'_3, t + t'_2\rangle)$.

$\rho_{check\_z_1=z_2-z_3}$ is a witness of $\psi_{check\_z_1=z_2-z_3}$ iff $t'_1 + t'_2 = 1$ (due to $x = 1$ transition from location h_2 to h_3) and $z_1 = t_{22} + t'_1 + t'_3 = 1$ and $z_2 = \beta + t + t'_3 = 1$ and $z_3 = t + t'_2 = 1$. Thus $\psi_{check\_z_1=z_2-z_3}$ holds iff $t_{22} = \beta$.

Lemma 22. *In the module check_x = z_1 in Figure 5.0.7, if the initial values were $z_1 = t_2$, $z_2 = 0$ and $x = t_4$ then $\psi_{check\_x=z_1}$ holds iff $t_2 = t_4$.*

Follows from simple calculations.

Lemma 23. *In the module check_3 ∗ x = 2 ∗ z_2 in Figure 5.0.7, if the initial values were $z_2 = \beta + t$ and $x = t$ then $\psi_{check\_3*x=2*z_2}$ holds iff $t = 2 * \beta$.*

Proof. Consider a path $\rho_{check\_3*x=2*z_2}$ from $(q_1, t, \langle 0, \beta + t, 0\rangle)$ to location H_F. $\rho_{check\_3*x=2*z_2} = (q_1, t, \langle 0, \beta + t, 0\rangle) \overset{\rho_{get^1_x}}{\dashrightarrow} (H_1, t', \langle 1 - t + t', \beta + t, t'\rangle)$. Then $\rho_{check\_3*x=2*z_2}$ is a witness of $\psi_{check\_3*x=2*z_2}$ iff $z_1 = 1 - t + t' = 1$ and $(H_1, t', \langle 0, \beta + t, t'\rangle) \models$ $\mathbf{E}\, \neg H_4\, \mathbf{U}\, \{H_4 \wedge \psi_{check\_z_1=z_2-z_3} \wedge z_2.\, \mathbf{E}(\neg H_F \wedge \psi_{check\_x=z_1})\, \mathbf{U}\, (H_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)\}$. From argument similar to Lemma 12, it is clear that $t' = t$.

Consider a path ρ_{H_4} from $(H_1, t, \langle 0, \beta + t, t\rangle)$ to location H_4. $\rho_{H_4} = (H_1, t, \langle 0, \beta + t, t\rangle) \longrightarrow (H_2, t, \langle 0, \beta + t, t\rangle) \xrightarrow{t_{22}} (H_2, t + t_{22}, \langle t_{22}, \beta + t, t\rangle)(H_3, t + t_{22}, \langle t_{22}, \beta + t, t\rangle) \xrightarrow{2-(t+t_{22})} (H_3, 2, \langle t_{22}, \beta + t, t\rangle) \xrightarrow{x=2?x:=0} (H_4, 0, \langle t_{22}, \beta + t, t\rangle)$. ρ_{H_4} is a witness of $\mathbf{E}\, \neg H_4\, \mathbf{U}\, \{H_4 \wedge \psi_{check\_z_1=z_2-z_3} \wedge z_2.\, \mathbf{E}(\neg H_F \wedge \psi_{check\_x=z_1})\, \mathbf{U}\, (H_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)\}$ iff $(H_4, 0, \langle t_{22}, \beta + t, t\rangle) \models \psi_{check\_z_1=z_2-z_3}$ and $(H_4, 0, \langle t_{22}, 0, t\rangle) \models \mathbf{E}(\neg H_F \wedge \psi_{check\_x=z_1})\, \mathbf{U}\, (H_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)$. From Lemma 22 we know that $t_{22} = \beta$.

Let ρ_{H_F} be a path from $(H_4, 0, \langle \beta, 0, t\rangle)$ to location H_F. $\rho_{H_F} = (H_4, 0, \langle \beta, 0, t\rangle) \xrightarrow{t_{24}} (H_4, t_{24}, \langle \beta, 0, t\rangle) \longrightarrow (H_5, t_{24}, \langle \beta, 0, t\rangle) \xrightarrow{t_{25}} (H_5, t_{24} + t_{25}, \langle \beta, t_{25}, t\rangle) \longrightarrow (H_6, t_{24} + t_{25}, \langle \beta, t_{25}, t\rangle) \xrightarrow{t_{26}} (H_6, t_{24} + t_{25} + t_{26}, \langle \beta, t_{25}, t + t_{26}\rangle) \xrightarrow{x=1?} (H_7, 1, \langle \beta, t_{25}, t + t_{26}\rangle) \xrightarrow{t_{27}} (H_7, 1 + t_{27}, \langle \beta + t_{27}, t_{25} + t_{27}, t + t_{26}\rangle) \longrightarrow (H_F, 1 + t_{27}, \langle \beta + t_{27}, t_{25} + t_{27}, t + t_{26}\rangle)$. This path is a witness of $\mathbf{E}(\neg H_F \wedge \psi_{check\_x=z_1})\, \mathbf{U}\, (H_F \wedge z_1 = 1 \wedge z_2 = 1 \wedge z_3 = 1)$ iff

1. $(H_5, t_{24}, \langle \beta, 0, t\rangle) \models \psi_{check\_x=z_1} \iff t_{24} = \beta$ from Lemma 22.
2. $t_{24} + t_{25} + t_{26} = 1$ and $z_1 = \beta + t_{27} = 1$ and $z_2 = t_{25} + t_{27} = 1$ and $z_3 = t + t_{26} = 1$.

Thus $\psi_{check\_x=z_1}$ holds iff $t = 2 * \beta$. $\square$

Now, we shall describe the final formula which will ensure that all the modules are associated with their respective formulae. Ψ :: $z_1.z_2.z_3.\mathbf{E}\ \psi_{all}\ \mathbf{U}\ (HALT\ \wedge\ z_3 = 0)$ where $\psi_{all} :: \bigwedge_{i=1,2}\ \psi_{I_i}\ \wedge\ \psi_{D_i}\ \wedge\ \psi_{Z_i}$.

Theorem 4. *If M is the two counter machine represented by $\mathcal{A}$ and Ψ then $\mathcal{A}, (l_1, 0, \langle 0, 0, 0\rangle) \models \Psi$ iff M halts.*

Proof. We show that if $\mathcal{A}, (l_1, 0, \langle 0, 0, 0\rangle) \models \Psi$ then M halts by building the halting computation of M.

Let ρ be the path witnessing Ψ. Then $\rho = (l_0, 0, \langle 0, 0, 0\rangle) \stackrel{x=0?}{\longrightarrow} \cdots \stackrel{x=0?}{\longrightarrow} (HALT, -, \langle -, -, 0\rangle)$ such that ψ_{all} holds in all the states.

From the construction we know that the sub-automaton starting at $(l_i, 0, \langle -, -, 0\rangle)$ simulates the instruction l_i in M. Let s be a state occuring in ρ.

- If $s = (l_i, 0, \langle -, -, 0\rangle)$ or $s = (l^j, -, \langle -, -, 0\rangle)$ then it trivially satisfies ψ_{all}.
- If $s = (I_1, t, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, -, 0\rangle)$ then it satifies ψ_{all} iff it satisfies ψ_{I_1}. From Lemma 11, we know that if $s \models \psi_{I_1}$ then $t = \frac{1}{2} * \frac{1}{2^{n_1} * 3^{n_2}}$. Thus n_1 has been incremented. Thus $s = (I_1, \frac{1}{2^{n_1+1} * 3^{n_2}}, \langle 1 - \frac{1}{2^{n_1+1} * 3^{n_2}}, -, 0\rangle)$.
- Similarly if $s = (D_1, t, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, -, 0\rangle)$ or $s = (I_2, t, \langle -, 1 - \frac{1}{2^{n_3} * 3^{n_4}} + t, 0\rangle)$ or $s = (D_2, t, \langle -, 1 - \frac{1}{2^{n_3} * 3^{n_4}} + t, 0\rangle)$, then n_2 or n_3 or n_4 have been incremented respectively.
- If $s = (l_1^1, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}}, -, 0\rangle)$ then $s \models \psi_{Z_1}$ and thus satisfies $\psi_{check\_n_1=n_2}$. From Lemma 18 it follows that $n_1 = n_2$. Similarly if $s = (l_1^2, 0, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}}, -, 0\rangle)$ then $n_1 \neq n_2$.

From the construction of $\mathcal{A}$, we know that locations labelled l_i coincide in the sub-automaton. Also, it is clear from $\mathcal{A}$ and Ψ that no time elpases in locations l_i, Ij, Dj, l_j^1 and l_j^2, $\forall i$ and $j \in \{1, 2\}$. Consider the case where $s = (I_1, t, \langle 1 - \frac{1}{2^{n_1} * 3^{n_2}} + t, -, 0\rangle)$ is a state in ρ. As no time elapses in I_1, there is a single instance of I_1 between two consecutive occurences of l_i and l_{i+1} and $s \models \psi_{I_1}$. Hence the state $(l_{i+1}, \frac{1}{2^{n_1+1} * 3^{n_2}}, \langle 1 - \frac{1}{2^{n_1+1} * 3^{n_2}}, -, 0\rangle)$ is reached after s. This ensures that values of stopwatch costs updated by the module $\mathcal{A}_{i+1}$ are indeed the result of the instruction l_i simulated by $\mathcal{A}_i$. Thus $\mathcal{A}$ and Ψ simulate M.

From the argument above it is clear that there is a single occurence of state $(l_i, 0, \langle -, -, -\rangle)$ in ρ, $\forall i$. Associate a tuple $(l_i, (n_1^i - n_2^i), (n_3^i - n_4^i))$ with each state $(l_i, 0, \langle 1 - \frac{1}{2^{n_1^i} * 3^{n_2^i}}, 1 - \frac{1}{2^{n_3^i} * 3^{n_4^i}}, 0\rangle)$ in ρ. Now the path ρ gives a sequence of tuples $(l_1, 0, 0)(l_2, -, -) \cdots (HALT, -, -)$. This sequence represents the halting computation of M.

To show that if M halts then $\mathcal{A}, (l_1, \langle 0, 0, 0\rangle) \models \Psi$ we construct a path ρ' witnessing Ψ using the halting computation of M.

Without loss of generality, let $(l_1, 0, 0)(l_2, 1, -) \cdots (l_i, C_1^i, C_2^i) \cdots (HALT, -, -)$ be the halting computation of M. Construct another sequence of tuples $(l_1, (0-0), (0-0))(l_2, (1-0), (0-0)) \cdots (l_i, (n_1^i - n_2^i), (n_3^i - n_4^i)) \cdots (HALT, -, -)$ such that $\forall i$, $n_1^i - n_2^i = C_1^i$ and $n_3^i - n_4^i = C_2^i$. Now construct a path ρ' in $\mathcal{A}$ such that $\rho' = (l_1, 0, \langle 0, 0, 0 \rangle) \dashrightarrow (I_1, \frac{1}{2}, \langle \frac{1}{2}, 0, 0 \rangle) \dashrightarrow (l_2, 0, \langle \frac{1}{2}, 0, 0 \rangle) \cdots (l_i, 0, \langle 1 - \frac{1}{2^{n_1^i} * 3^{n_2^i}}, 1 - \frac{1}{2^{n_3^i} * 3^{n_4^i}}, 0 \rangle) \cdots (HALT, -, \langle -, -, 0 \rangle)$. Additionally, if the instruction l_i corresponds to checking if $C_1 = 0$ then select state $(l_1^1, 0, \langle -, -, - \rangle)$ after $(l_i, 0, \langle -, -, - \rangle)$ if $C_1^i = n_1^i - n_2^i = 0$ else choose $(l_1^2, 0, \langle -, -, - \rangle)$.

It is clear from ρ' that no time elpases in locations l_i, Ij, Dj, l_j^1 and l_j^2, $\forall i$ and $j \in \{1, 2\}$. From Lemma 11 it follows that $\forall i$, $(I_1, \frac{1}{2^{n_1^i+1} * 3^{n_2^i}}, \langle 1 - \frac{1}{2^{n_1^i+1} * 3^{n_2^i}}, -, 0 \rangle) \models \psi_{I_1}$. Similar argument holds for locations I_2, D_1, D_2. Also from Lemma 18 if $C_1^i = 0$ then $(l_1^1, 0, \langle -, -, - \rangle) \models \psi_{Z_1}$ else $(l_1^2, 0, \langle -, -, - \rangle) \models \neg \psi_{Z_1}$. Thus ρ' is a witness of Ψ as ψ_{all} holds in all states in ρ'.

Thus $\mathcal{A}, (l_1, 0, \langle 0, 0, 0 \rangle) \models \Psi$ iff M halts. □

Theorem 5. *The $WCTL_{2r}$ modelchecking problem on WIRTAs with 1 clock and 3 stopwatch costs is undecidable.*

This result follows from the construction of $\mathcal{A}$, $WCTL_{2r}$ formula Ψ and Theorem 4.

6 Conclusion

In this report, we have studied of the WCTL model checking problem on WTAs. We have identified a subclass of WTAs on which $WCTL_1$ model checking is decidable, irrespective of the number of clocks and costs. The question for $WCTL_{2r}$ is undecidable with WIRTAs having 3 stopwatch costs and 1 clock. Some interesting questions that need to be answered are (i) The decidability question for model checking $WCTL_2$ on WIRTA (ii) Model checking $WCTL_1$ with multi-constrained modalities.

6.1 Future work

We propose to continue the study of model checking with the following open problems.

1. $WCTL_2$ over WIRTA.
2. $WCTL_{2r}$ over WIRTA with costs.
3. $WCTL_1$ with multi constrained modalities.
4. Reducing the number of stopwatches in the undecidability result.

Additionally, we would like to investigate other interesting subclasses of WTA and variations of WCTL.

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