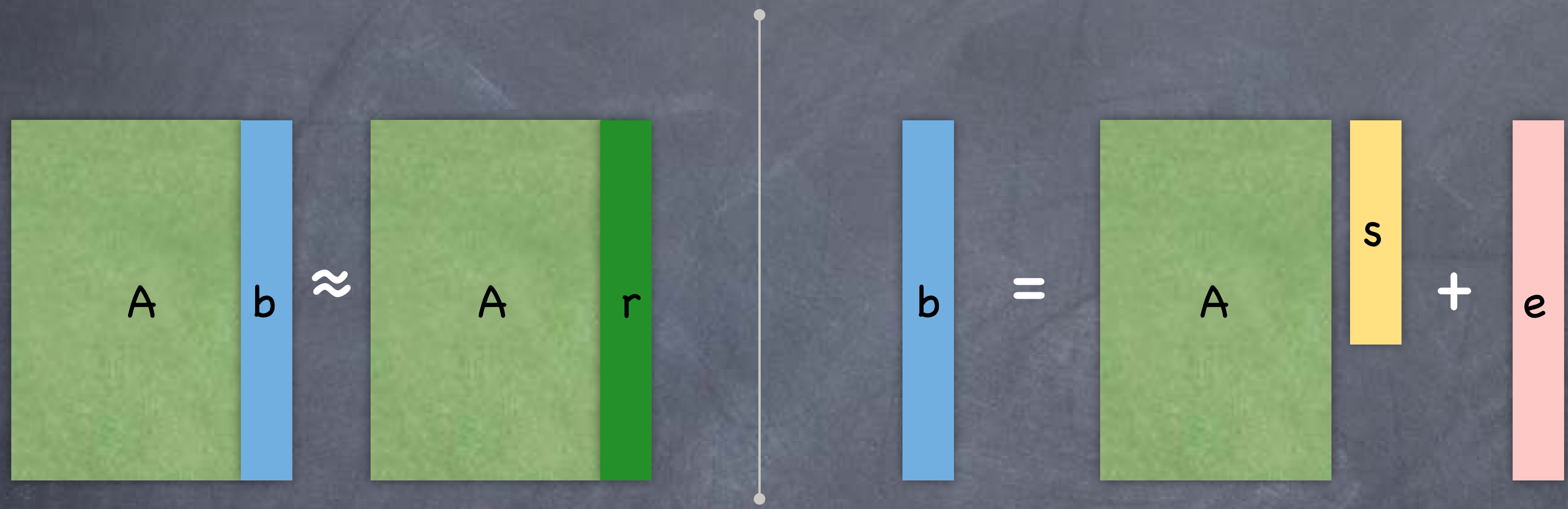
Lattice Cryptography: Towards Fully Homomorphic Encryption

Lecture 20

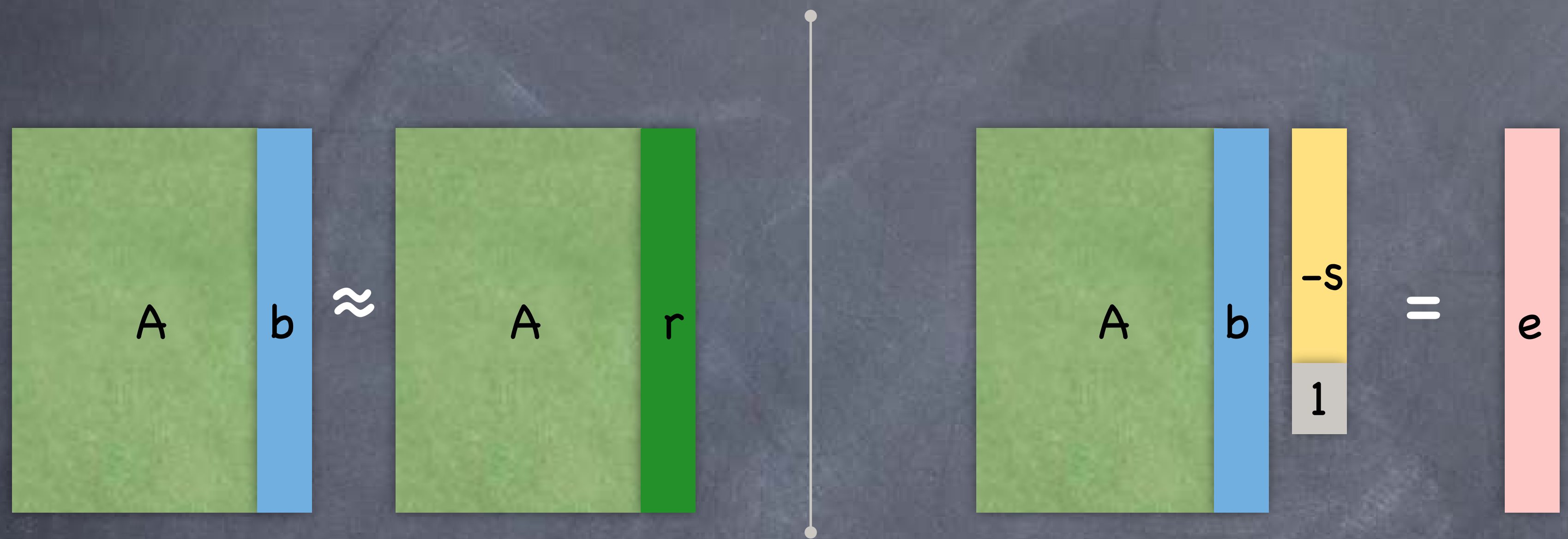
Recall

Learning With Errors



- LWE (decision version): $(A, A\underline{s}+\underline{e}) \approx (A, \underline{r})$, where A random matrix in $A \in \mathbb{Z}_q^{m \times n}$, $\underline{s}$ uniform, $\underline{e}$ has "small" entries from a Gaussian distribution, and $\underline{r}$ uniform.
- Average-case solution for LWE $\Rightarrow$ Worst-case solution for GapSVP (for appropriate choice of parameters)

Learning With Errors



- LWE (decision version): $(A, A\underline{s}+\underline{e}) \approx (A, \underline{r})$, where A random matrix in $A \in \mathbb{Z}_q^{m\times n}$, $\underline{s}$ uniform, $\underline{e}$ has "small" entries from a Gaussian distribution, and $\underline{r}$ uniform.
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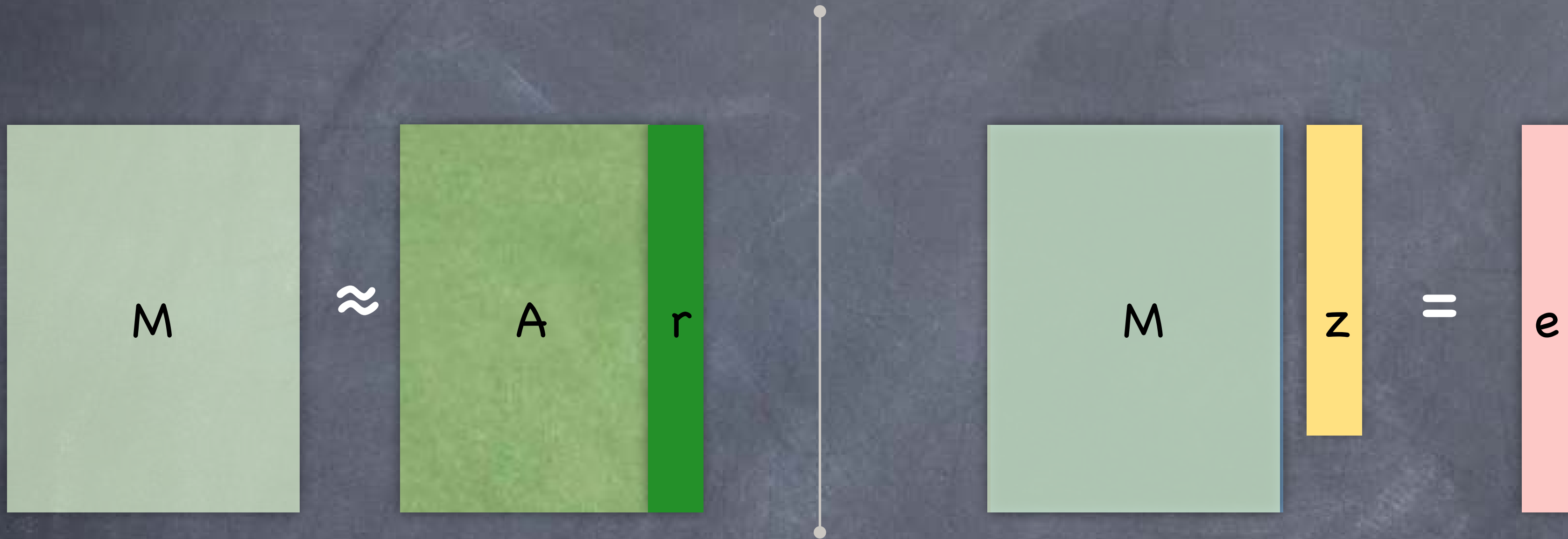
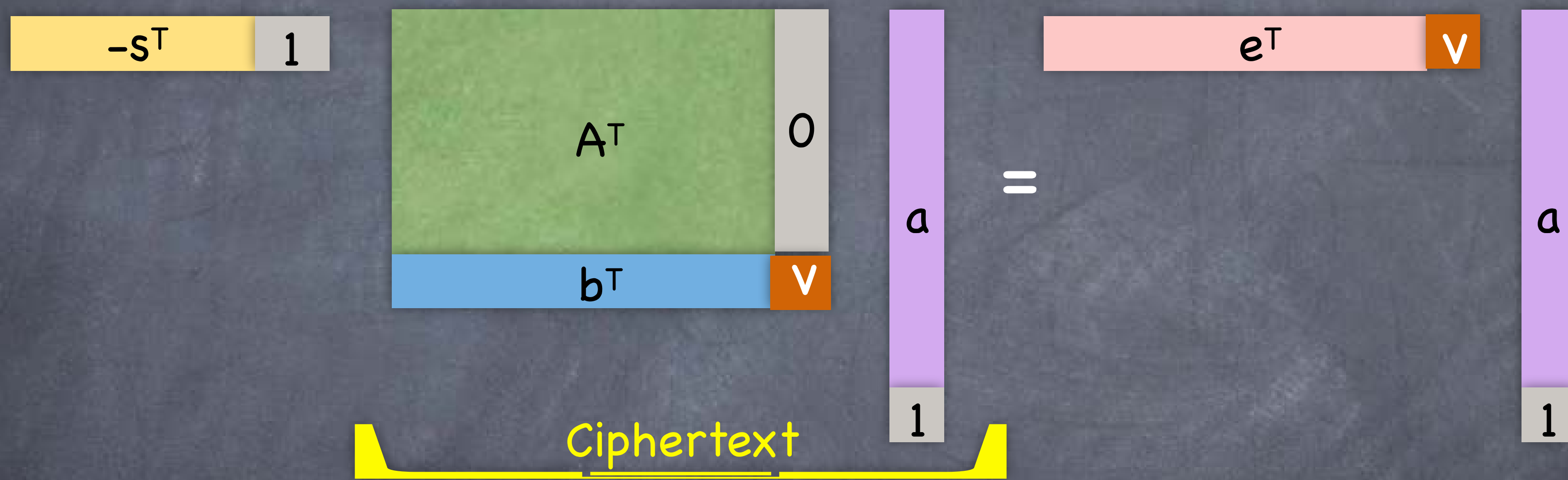


- i.e., a pseudorandom matrix $M \in \mathbb{Z}_q^{m\times n'}$ and $\underline{\mathbf{z}} \in \mathbb{Z}_q^{n'}$ s.t. entries of $M\underline{\mathbf{z}}$ are all small ($n'=n+1$)

Recall

PKE from LWE



- Ciphetext = $[M^T|\underline{\mathbf{m}}]\ \underline{\mathbf{a}}$ where $\underline{\mathbf{m}}$ encodes the message, $\mathbf{a} \in \{0,1\}^m$
- Decryptng: From $\underline{\mathbf{z}}^T[M^T|\underline{\mathbf{m}}]\underline{\mathbf{a}} = \underline{\mathbf{e}}^T\underline{\mathbf{a}} + \underline{\mathbf{z}}^T\underline{\mathbf{m}}$ where $\underline{\mathbf{e}}^T\underline{\mathbf{a}}$ is small. Encoding should allow decoding from this.
- CPA security: $M^T\underline{\mathbf{a}}$ is pseudorandom
  - Claim: If $M \in \mathbb{Z}_q^{m\times n'}$ is uniform, $\mathbf{a} \in \{0,1\}^m$, and $m >> n' \log q$, then $M^T\underline{\mathbf{a}}$ is very close to being uniform

Randomness Extraction

- Entries in $\underline{\mathbf{a}}$ are not uniformly random over $\mathbb{Z}_q^m$, but concentrated on a small subset $\{0,1\}^m$. We need $M^T\underline{\mathbf{a}}$ to be uniform over $\mathbb{Z}_q^{n'}$
- Follows from two more generally useful facts:
  - $H_M(\underline{\mathbf{a}}) = M^T\underline{\mathbf{a}}$ is a 2-Universal Hash Function (for non-zero $\underline{\mathbf{a}}$)
  - If H is a 2-UHF, then it is a good **randomness extractor**
    - If $m \gg n' \log q$, the entropy of $\underline{\mathbf{a}}$ (m bits) is significantly more than that of a uniform vector in $\mathbb{Z}_q^{n'}$ and a good randomness extractor will produce an almost uniform output

Universal Hashing

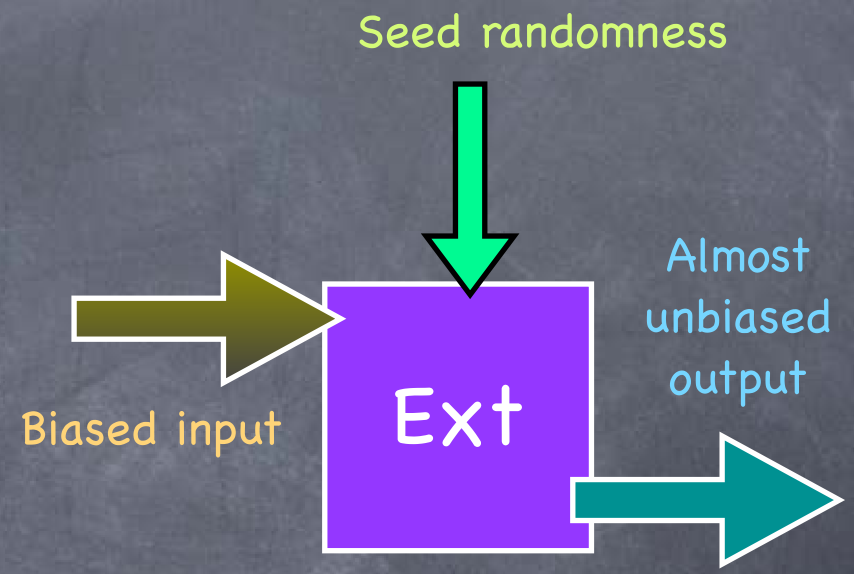
- Combinatorial HF: $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$. $h(x)=h(y)$ w.n.p
- Even better: 2-Universal Hash Functions
  - "Uniform" and "Pairwise-independent"
  - $\forall x,z\ \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=z\ ] = 1/|Z|$ (where $h:X \rightarrow Z$)
  - $\forall x \neq y,w,z\ \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=w,\ h(y)=z\ ] = 1/|Z|^2$
    - $\Rightarrow\ \forall x \neq y\ \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=h(y)\ ] = 1/|Z|$

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$
0	0	0	1	1
1	0	1	0	1
2	1	0	0	1

Negligible collision-probability if super-polynomial-sized range

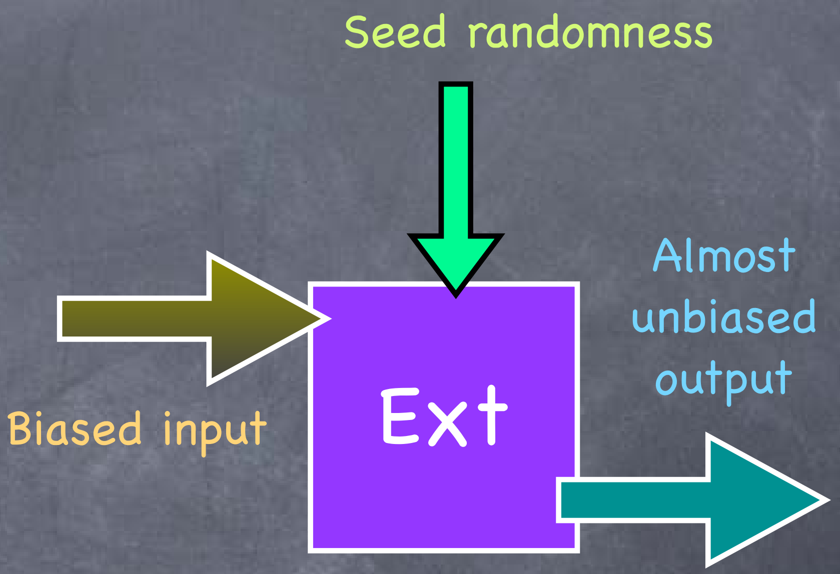
- e.g. $h_{a,b}(x) = ax+b$ (in a finite field, $X=Z$)
  - $\Pr_{a,b}[\ ax+b = z\ ] = \Pr_{a,b}[\ b = z-ax\ ] = 1/|Z|$
  - $\Pr_{a,b}[\ ax+b = w,\ ay+b = z] = ?$ Exactly one (a,b) satisfying the two equations (for $x \neq y$)
    - $\Pr_{a,b}[\ ax+b = w,\ ay+b = z] = 1/|Z|^2$
- Exercise: $M\underline{\mathbf{x}}$ (M random matrix) is a 2-UHF for non-zero vectors $\underline{\mathbf{x}}$

Randomness Extractor



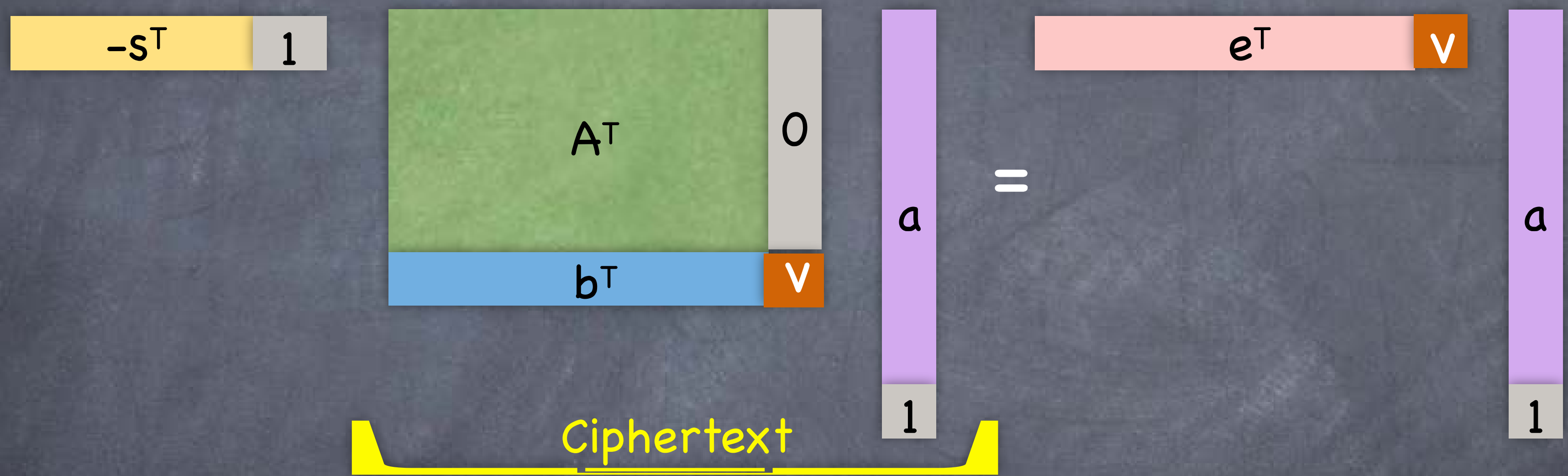
- Input has high "min-entropy"
  - i.e., probability of any particular input string is very low
- Seed uniform and independent of input
- Output vector is shorter than the input
- Ext(inp,seed)) ≈ Uniform
  - Statistical closeness
- A **strong extractor**: (seed, Ext(inp,seed)) ≈ (seed,Uniform)
  - i.e., for any input distribution, most choices of seed yield a good deterministic extractor

Randomness Extractor



- Leftover Hash Lemma:
  - Any 2-UHF is a strong extractor that can extract almost all of the min-entropy in the input
- A very useful result
  - Much stronger than what we need today:
    - Only for a particular 2-UHF ($H_M(\underline{\mathbf{x}}) = M\underline{\mathbf{x}}$)
    - Only for a particular input distribution ($\underline{\mathbf{x}}$ uniform over $\{0,1\}^m$)

PKE from LWE



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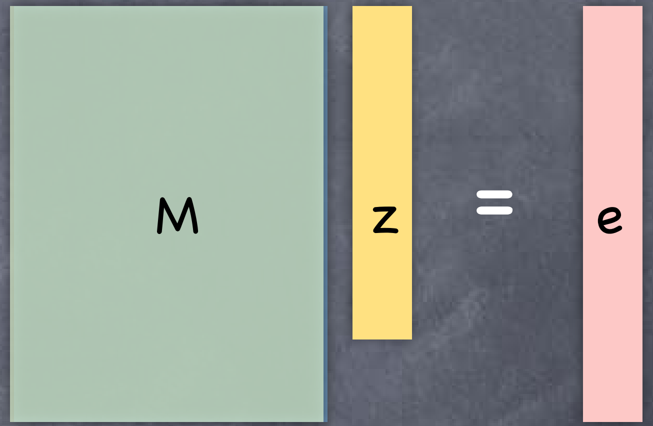
Gentry-Sahai-Waters

- Want to allow homomorphic operations on the ciphertext
- Rough plan: Ciphertext is a matrix. Addition and multiplication of messages by addition and multiplication of ciphertexts
- Recall from LWE: $M \in \mathbb{Z}_q^{m \times n}$ and $\underline{\mathbf{z}} \in \mathbb{Z}_q^{n}$

  s.t. $\underline{\mathbf{z}}^T M^T$ has small entries



- First attempt: Public-Key = M, Secret-key = $\underline{\mathbf{z}}$
  - $Enc(\mu) = M^T R + \mu I$ where $\mu \in \{0,1\}$, $R \leftarrow \{0,1\}^{m \times m}$, and $I_{m \times m}$ identity
  - Security: LWE (and LHL) $\Rightarrow$ $M^T R$ is pseudorandom
  - $Dec_z(C)$: $z^T C = e^T R + \mu z^T$ has "error" $\underline{\delta}^T = e^T R$. Can recover m since error has small entries (w.h.p.)

Gentry-Sahai-Waters

- First attempt:
  - $\mathrm{Enc}(\mu) = M^T R + \mu I$
  - $\mathrm{Dec}_z(C)$: $z^T C = e^T R + \mu z^T$ has error $\underline{\delta}^T = e^T R$
  - $C_1 + C_2 = M^T(R_1 + R_2) + (\mu_1 + \mu_2) I$ has error $\underline{\delta}^T = \underline{\delta}_1^T + \underline{\delta}_2^T$
    - Error adds up with each operation
    - OK if there is an a priori bound on the depth of computation: Levelled Homomorphic Encryption (a.k.a. Somewhat HE)
  - $C_1 \times C_2$: Error = ?
    - $z^T C_1 C_2 = (\underline{\delta}_1^T + \mu_1 z^T) C_2 = \underline{\delta}_1^T C_2 + \mu_1(\underline{\delta}_2^T + \mu_2 z^T)$
    - Error = $\underline{\delta}_1^T C_2 + \mu_1 \underline{\delta}_2^T$
    - Problem: Entries in $\underline{\delta}_1^T C$ may not be small! (Since $\mu_1 \in \{0,1\}$ the other vector has small entries)

Gentry-Sahai-Waters

- Problem: Entries in $\underline{\delta}_1^T C_2$ may not be small
- Solution Idea: Represent ciphertext as bits!
  - But homomorphic operations will be affected
  - Observation: Reconstructing a number from bits is a linear operation
  - If $\alpha \in \mathbb{Z}_q^m$ has bit-representation $B(\alpha) \in \{0,1\}^{km}$ $(k=O(\log q))$, then $G\,B(\alpha) = \alpha$, where $G \in \mathbb{Z}_q^{m\times km}$ (all operations in $\mathbb{Z}_q$)
    - B can be applied to matrices also as $B : \mathbb{Z}_q^{m\times n} \rightarrow \mathbb{Z}_q^{km\times n}$ and we have $G\,B(\alpha) = \alpha$

Gentry-Sahai-Waters

- The actual scheme:
  - Will only support messages $\mu \in \{0,1\}$ and NAND operations (could support addition mod q too, but not mod 2) up to an a priori bounded depth
  - Public key $M \in \mathbb{Z}_q^{m\times n}$. Private key $\underline{\mathbf{z}}$ s.t. $\underline{\mathbf{z}}^T M^T$ has small entries.
  - $\mathrm{Enc}(\mu) = M^T R + \mu G$ where $R \leftarrow \{0,1\}^{m\times km}$ (and $G \in \mathbb{Z}_q^{m\times km}$ the matrix to reverse bit-decomposition)
  - $\mathrm{Dec}_z(C)$: $\underline{\mathbf{z}}^T C = \underline{\delta}^T + \mu \underline{\mathbf{z}}^T G$ where $\underline{\delta}^T = e^T R$
  - $\mathrm{NAND}(C_1,C_2)$: $G - C_1 \cdot B(C_2)$ (G is a (non-random) encryption of 1)
    - $\underline{\mathbf{z}}^T C_1 \cdot B(C_2) = \underline{\mathbf{z}}^T C_1 \cdot B(C_2) = (\underline{\delta}_1^T + \mu_1 \underline{\mathbf{z}}^T G)\, B(C_2)$

$$= \underline{\delta}_1^T B(C_2) + \mu_1 \underline{\mathbf{z}}^T C_2 = \underline{\delta}^T + \mu_1\mu_2 \underline{\mathbf{z}}^T G$$

where $\underline{\delta}^T = \underline{\delta}_1^T B(C_2) + \mu_1 \underline{\delta}_2^T$ has small entries