Composition and Iteration for SNARKs

Lecture 23

Composition

- Outsourcing the verification back to the prover
  - Proof for statement σ: Proof π_1 that there exists a proof π_0 for σ which will be accepted by a verifier V_0
- Can make the proof shorter and the verification faster
  - Prover's work increases, but not by much if π_0 short & V_0 fast
- Can make the overall proof ZK if the outer proof π_1 is ZK

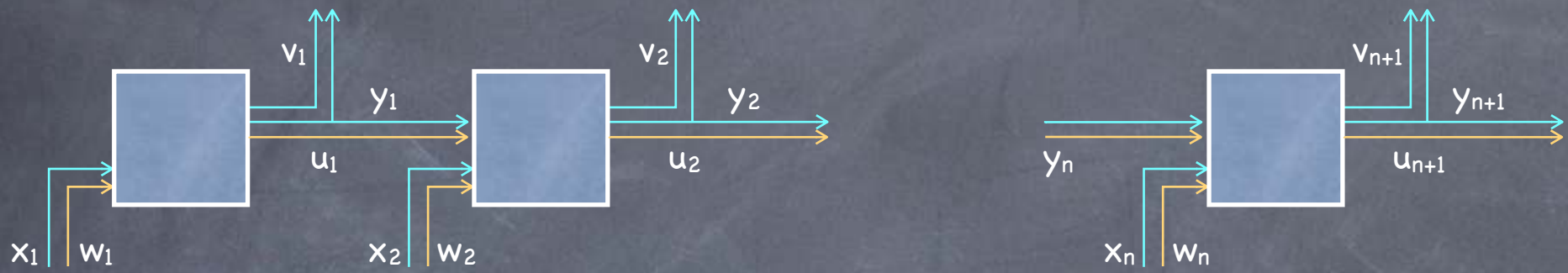
Composition

- Overheads/limiting factors
  - Once the witness is short enough, succinctness not possible
  - No more a secure scheme in ROM even if π_0,π_1 were: Hash function used in π_0 is implemented as a hash function
  - Knowledge soundness degrades: For a time t adversary, Ext_1 runs in time $poly_1(t)$ to extract π_0. Ext_0 extracts witness from Ext_1 in time $poly_0(poly_1(t))$.
  - Suppose σ is for circuits over a field $F_{(0)}$ and the computation of V_0 is over a field $F_{(1)}$, then π_1 should support statements over $F_{(1)}$. May limit choice of the underlying proof systems.

Composition

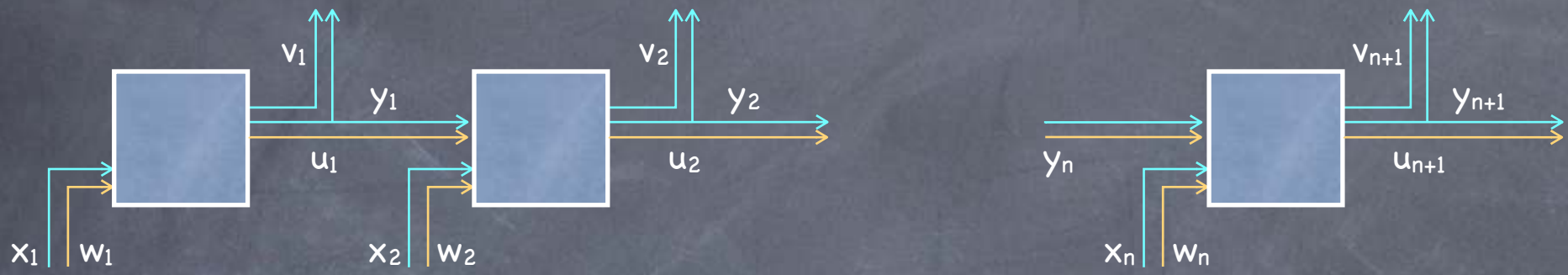
- Proof for statement σ: Proof π_1 that there exists a proof π_0 for σ which will be accepted by a verifier V_0
- Suppose σ over a field $F_{(0)}$ and verifying π_0 uses computations over a field $F_{(1)}$, then π_1 should support statements over $F_{(1)}$
- Recall: when using discrete-log based polynomial commitment schemes, the field for the polynomial is $\mathbb{F}_p$, where p is the order of the cyclic group where discrete log is assumed to be hard
  - The most efficient candidates for such groups are elliptic curve groups. Such a group consists of a set of p points of the form $(x,y) \in \mathbb{F}^2$ for some other "base field" $\mathbb{F}$, and the group operations use field operations over $\mathbb{F}$. Here $F_{(0)}$ is $\mathbb{F}_p$, $F_{(1)}$ is $\mathbb{F}$.
- Verification of π_1 will be over another field $F_{(2)}$. For deeper composition a proof system for statements in $F_{(2)}$ needed.
  - E.g., From 2 groups of order p, q, with base fields $\mathbb{F}_q$, $\mathbb{F}_p$, resp.

Iterative Statements



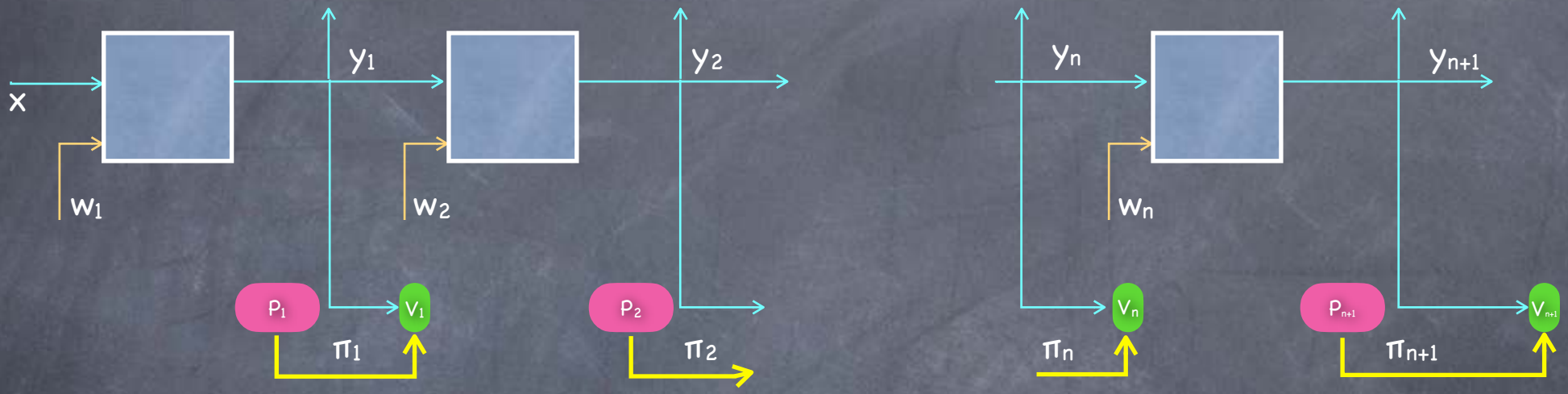
- Many computations encountered in applications are iterative in nature
  - e.g., evaluating a Merkle-Damgard iterated hash function
- Goals:
  - To reduce prover's total effort
    - Linear in the number of steps
  - To allow different provers to carry out different steps

Iterative Statements



- When propagating proof
  - Omit x_i (except x_1) as verifier cannot take all x_i directly as input. Can instead include a Merkle hash of all x_i in x_1, and w_i can include the opening to x_i
  - Omit u_i by instead including a hiding commitment of u_i in y_i, and opening it in w_{i+1}
  - Omit v_i and include it in y_i as all of public output needs to go into the next stage's verifier

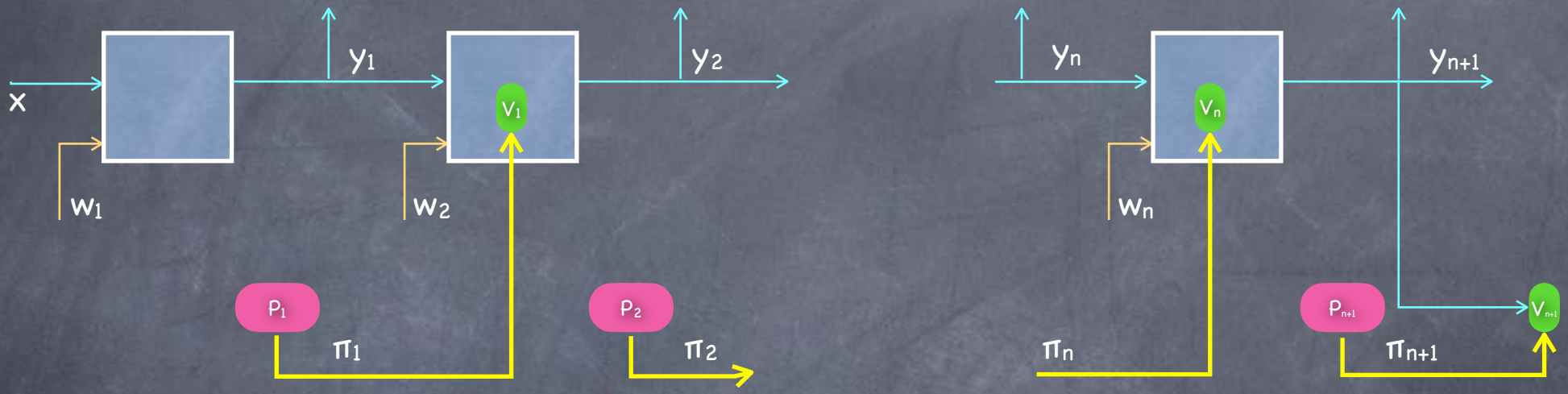
Iterative Statements



- Naïve scheme: At each step, give a proof that the computation in the last step is consistent with the previous step's output
  - Allows incremental proving/verification: can forget witnesses/proofs from previous steps
  - Prover's time is linear in n (if each step of computation is taken to be of constant time)
  - But overall proof is long, and verification takes time proportional to n

Iterative Statements

With Composition

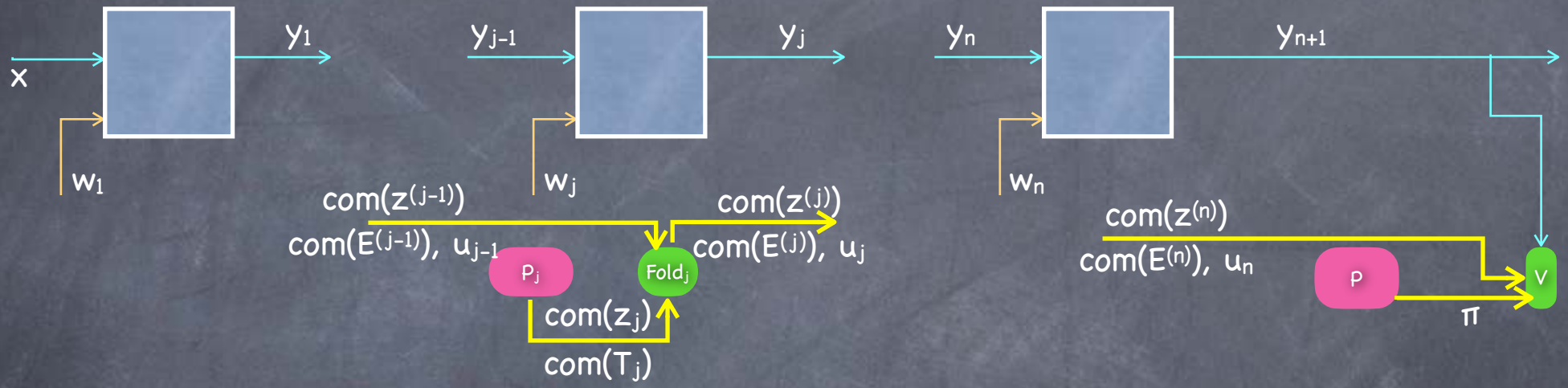


- Idea: Move verification into the iterated computation
  - Statement proven by P_{i+1} includes that V_i accepted its proof
- Note: Cannot prove soundness of the overall scheme in the Random Oracle Model, since the hash function modelled as the RO is now part of the computation
- Note: Provable knowledge soundness degrades exponentially with number of steps. May be directly assumed as a heuristic.
- Prover's time and space requirements are still linear in the number of steps. But each step now has a larger computation.

Folding

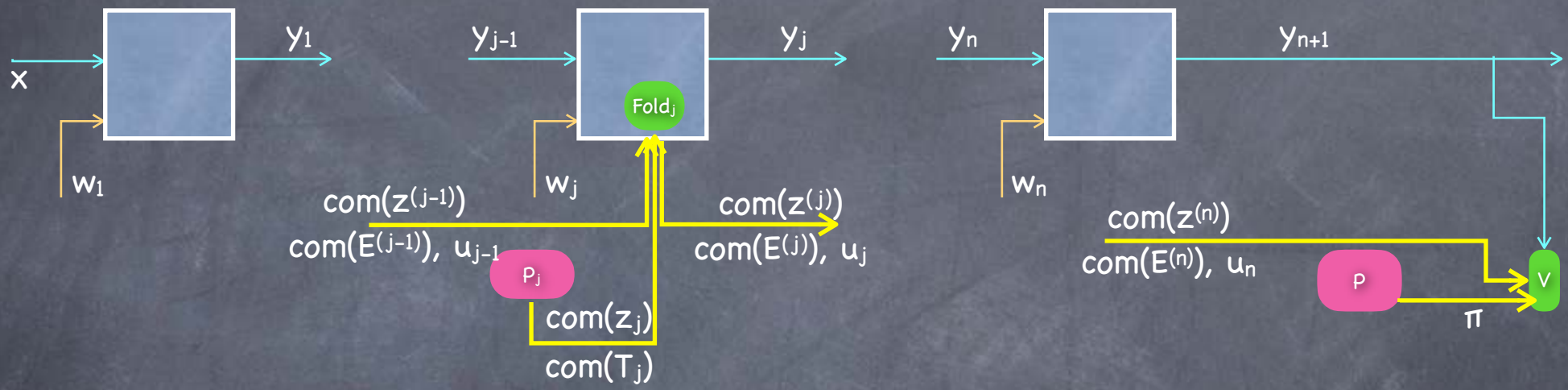
- A technique for realising iteration more efficiently for the prover
  - Will use R1CS representation of F
    - $F(y_{j-1}) = y_j$ iff $\exists z=(y_{j-1},y_j,w)$ s.t. $Az \circ Bz = Cz$
- Idea: Combine two R1CS instances (with same A,B,C) into a single R1CS instance, so that the latter can be satisfied only if the original instances can both be (with high probability)
- Uses a more general representation: "Extended" R1CS
  - $Az \circ Bz = u\,Cz + E$, where $u \in \mathbb{F}$ and $E \in \mathbb{F}^m$ is a vector
- Given instances $Az_1 \circ Bz_1 = u_1Cz_1 + E_1$ and $Az_2 \circ Bz_2 = u_2Cz_2 + E_2$, define instance $Az \circ Bz = uCz + E$, where $u = u_1+ru_2$ and E is such that $z=z_1+rz_2$ satisfies
  - $E = E_1 + r^2E_2 + rT$, where T has the "cross terms":
    $T = Az_1 \circ Bz_2 + Az_2 \circ Bz_1 - u_1Cz_2 - u_2Cz_1$
- For any row, $\langle a_i,z_1+rz_2\rangle \langle b_i,z_1+rz_2\rangle = (u_1+ru_2)\langle c_i,z_1+rz_2\rangle + E_{1i} + r^2E_{2i} + rT_i$ holds for at most two values of r, unless all 3 coefficients of r are 0: requiring the three conditions above to hold for that row

Folding for Iteration



- Expanded R1CS representation of F:
  $F(y_{j-1}) = y_j$ iff $\exists z=(y_{j-1},y_j,w_j)$ s.t. $Az \circ Bz = uCz + E$, where $u=1$, $E=0$
- At each round, the prover commits (via homomorphic commitment) to w_j and y_j to complete the commitment of $z_j = (y_{j-1},y_j,w_j)$. Also to T_j.
- "Folder" picks r_j and computes commitment of $z^{(j)} = z^{(j-1)} + r_j z_j$. Also of $E^{(j)} = E^{(j-1)} + r_j^2 E + r_j T_j = E^{(j-1)} + r_j T_j$ (since $E=0$). Let $u^{(j)} = u^{(j-1)} + r_j$.
- Finally prover gives a SNARK for ER1CS with $E^{(n)}$, $z^{(n)}$ (committed), $u^{(n)}$,y_{n+1}
  - Adapting R1CS SNARKs that use homomorphic commitments to ER1CS

Folding for Iteration



- Nova: Convert to a SNARK, by moving folders into the computation represented by the ER1CS
- Only one verification at the end
  - Verifier may be interested in $(y_1,\ldots,y_{n+1})$, not just y_{n+1}
  - Fix: The output of the computation will be $y'_j = (y_j, y^{(j)})$ where $y^{(j)} = \text{Hash}(y_j, y^{(j-1)})$. In addition to the above proof, prover can send $y_1,\ldots,y_{n+1}$, so that $y^{(n+1)}$ can be checked.
- Note: As before, provable knowledge soundness degrades exponentially with number of steps. May be directly assumed as a heuristic.