Miscellany

Lecture 25

Using iO: Examples

Shallow Computation: Why and How

Using iO: An Example

- PKE from SKE using iO
  - PK = iO($f_K(\cdot)$) where $f_K(s,m) = (PRG(s), PRF_K(PRG(s)) \oplus m)$
  - Problem using iO: iO may not hide K!
  - But the functionality of f_K depends only on PRF_K evaluated on the range of PRG. So it is plausible that there are alternate representations of f_K that does not reveal K fully
  - Idea: Imagine challenge ciphertext is $(r, PRF_K(r) \oplus m)$ where r is not in the range of PRG!
    - Cannot tell the difference by security of PRG
    - Revealing functionality f_K need not reveal $PRF_K(r)$

Using iO: An Example

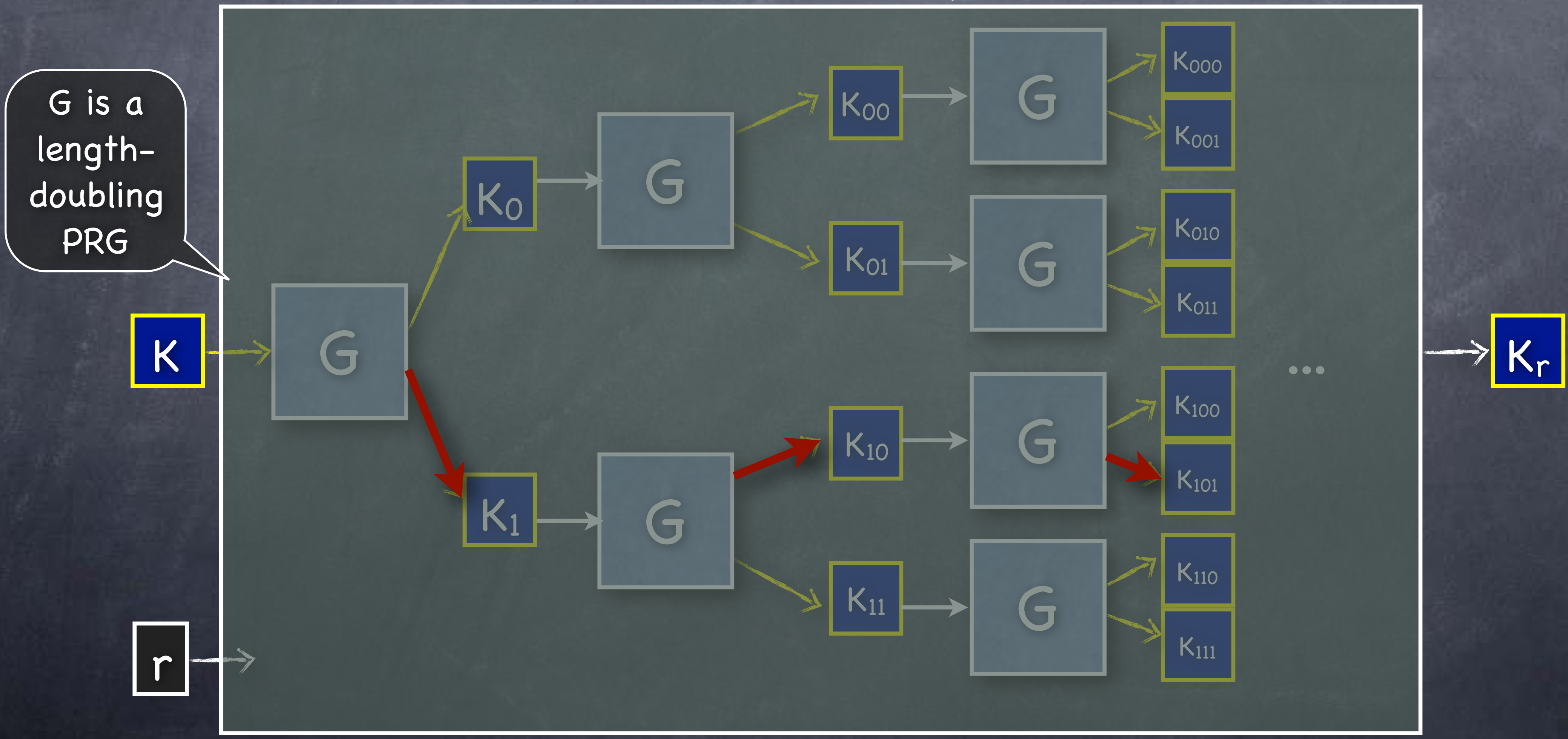
Punctured PRF used only in proof

By modifying the standard construction

- PKE from SKE using iO
  - PK = iO($f_K(\cdot)$) where $f_K(s,m) = (PRG(s), PRF_K(PRG(s)) \oplus m)$
  - Idea: Imagine challenge ciphertext is $CT' = (r, PRF_K(r) \oplus m)$ where r is not in the range of PRG!
    - Cannot tell the difference with real CT by security of PRG
  - Punctured PRF: Key $K^{\bar{r}}$ to evaluate PRF_K on inputs other than r, such that $PRF_K(r)$ is pseudorandom given $K^{\bar{r}}$
  - $f'_{K^{\bar{r}}}(s,m) = (PRG(s), PRF'_{K^{\bar{r}}}(PRG(s)) \oplus m)$, is functionally equivalent to f_K, where PRF′ is the PRF punctured at input r
  - Let $PK' = iO(f'_{K^{\bar{r}}}(\cdot))$. Then $(CT,PK) \approx (CT',PK')$
    - (CT',PK') completely hides m, even if PK′ revealed all of $K^{\bar{r}}$

Pseudorandom Function (PRF)

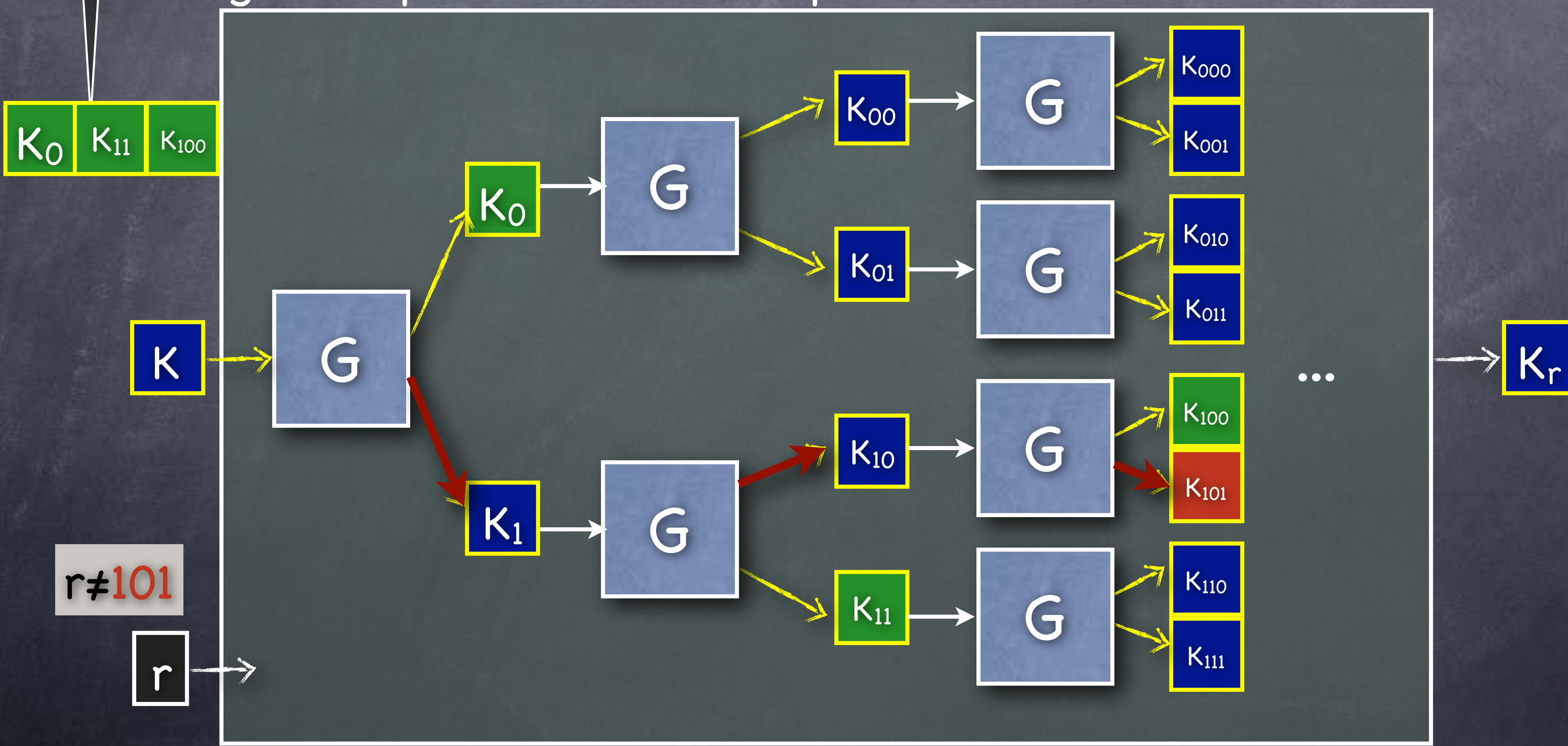
- A PRF can be constructed from any PRG



Pseudorandom Function (PRF)

Punctured Key: $K^{\overline{101}}$

- e.g., PRF punctured at an input 101:



Circuit Depth

- Functions f: $\{0,1\}^* \to \{0,1\}^*$ are often represented as circuit families (boolean or arithmetic)
  - Family of circuits **C** = $\{ C^n \}_{n\geq 1}$
    - Each circuit is a DAG, with n input wires. Will restrict ourselves to circuits with 2-input gates
    - For each input size n there is a separate circuit C^n (w.l.o.g., same output size for each fixed input size)
- Depth of a DAG: length of the longest root-to-leaf path
- **C** said to have "constant depth" if depth(C^n) $\leq$ c, for all n
- **C** in class NC^i if depth(C^n) $\leq c \cdot \log^i n$, for some c
  - Note: In NC^0 circuits each output wire connected to a constant number of input wires

Recall

Bootstrapping for iO

- iO candidate from multi-linear map candidates, using matrix programs
  - Polynomial sized iO if polynomial-sized matrix programs
  - Barrington's Theorem: NC^1 functions have polynomial-sized matrix programs (with 5x5 matrices)
  - Can "bootstrap" from this to all polynomial-sized circuits/ polynomial-time computable functions, assuming Fully Homomorphic Encryption with decryption in NC^1

Bootstrapping for iO

- Idea: Carry out FHE (for polynomial depth) evaluation, and use obfuscated program to do decryption
  - Function C will be encrypted, input m can be given in the clear
  - Let U denote a (deep) circuit s.t. U(C,m) = C(m). Let U_m be U with m hardwired as the second input.
  - Obfuscation: (σ,π) where σ=FHE-Enc(C) and π=iO(P) where P is a low-depth program that decrypts an FHE ciphertext σ^*, but only if it is obtained by evaluating U_m homomorphically on σ (for some input m)
    - How can P ensure this without computing U_m itself?
    - P takes a proof that $\sigma^* = F(m') :=$ FHE-Eval($U_{m'}$,σ) for some m'
      - Proof: σ^* and all wire values in circuit evaluating F(m'). Can verify each gate separately (in NC^0), and AND the results (in NC^1) to get the full verification result

Bootstrapping for iO

- Obfuscation: (PK,σ,π) where σ=FHE-Enc$_{PK}(C)$ and π=iO(P)
  - $P(\sigma^*,\varphi)$ = FHE-Dec$_{SK}(\sigma^*)$ if Verify(σ^*,φ)=1
  - Proof φ is for the claim: $\exists$ m′ s.t. σ^* = FHE-Eval$_{PK}(U_{m'},\sigma)$
- Evaluation: Compute σ^* and φ using m. Run $\pi(\sigma^*,\varphi)$ to get C(m)
- Secure? Need to hide representation of C
- But π may not hide the FHE decryption key SK!
- Idea: Have multiple representations of P s.t. some representations don't reveal SK or anything beyond C's functionality
- Will have $\sigma=(\sigma_1,\sigma_2)$, with $\sigma_i \leftarrow$ FHE-Enc$_{PK_i}(C)$. And the claim proven is $\exists$ m′ s.t. σ_1^* = FHE-Eval$_{PK_1}(U_{m'},\sigma_1) \wedge \sigma_2^*$ = FHE-Eval$_{PK_2}(U_{m'},\sigma_2)$

Bootstrapping for iO

- Obfuscation: $(PK_1,PK_2,\sigma_1,\sigma_2,\pi)$ where $\sigma_i \leftarrow$ FHE-Enc$_{PK_i}(C)$ and π=iO(P_1)
  - $P_1(\sigma_1^*,\sigma_2^*,\varphi)$ = FHE-Dec$_{SK_1}(\sigma_1^*)$ if Verify$(\sigma_1^*,\sigma_2^*,\varphi)$=1
  - Proof φ for claim $\underline{\exists\ m'\text{ s.t. for } i=1,2,\ \sigma_i^* = \text{FHE-Eval}_{PK_i}(U_{m'},\sigma_1)}$
- Evaluation: Compute σ_1^*,σ_2^*, φ using m. Run $\pi(\sigma_1^*,\sigma_2^*,\varphi)$ to get C(m)
- Consider functionally equivalent C_1 and C_2 and following "hybrids"
- 1. Obfuscation of C_1 : $\sigma_i \leftarrow$ FHE-Enc$_{PK_i}(C_1)$ and π=iO(P_1)
- 2. Uses $\sigma_i \leftarrow$ FHE-Enc$_{PK_i}(C_i)$

(1) ≈ (2): FHE security for SK_2

(2) ≈ (3): By iO. P_1, P_2 functionally equivalent!

- 3. Uses π=iO(P_2) where P_2 uses SK_2 to decrypt σ_2^*

(3) ≈ (4): FHE security for SK_1

- 4. Uses $\sigma_i \leftarrow$ FHE-Enc$_{PK_i}(C_2)$

(4) ≈ (5): Again by iO.

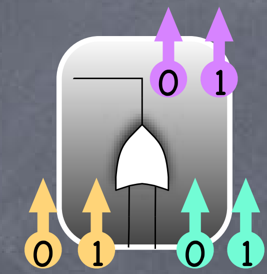
- 5. Uses π=iO(P_1). This is an honest obfuscation of C_2.

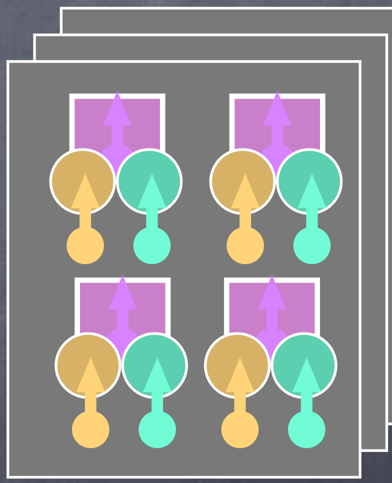
Depth and Interaction

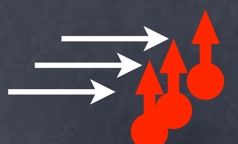
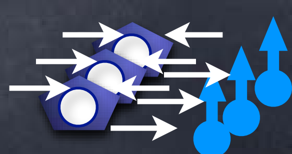
- Recall the GMW and BGW protocols
- Gate-by-gate evaluation of a circuit (DAG)
- Gates can be evaluated in any order as long as we respect a topological sort
- Can parallelise by grouping gates into levels
  - Number of rounds of interaction = number of levels
  - Smallest number of levels = depth of the circuit
- Moral: Functions with shallow circuits are quicker to evaluate
- Can sometimes do better by working with low-depth "**randomized encoding**" of functions than directly with their own circuits
  - Coming up: An example of randomized encoding

Garbled Circuits

- Recall: Each wire w has two keys ($K_{w=0}$ and $K_{w=1}$). Each garbled gate has 4 boxes with keys for the output wire, locked with keys for input wires
  - Locking: $\mathrm{Enc}_{K_{x=a}}(\mathrm{Enc}_{K_{y=b}}(K_{w=g(a,b)}))$
- Randomized Encoding of C(x):
  - { Garbled gates for C, Keys for input x}
- Reveals nothing but C(x) (only computationally secure)
- Decoding has depth proportional to the circuit C
- But encoding depth independent of C!
  - Pick all keys, and all garbled gates can be prepared in parallel



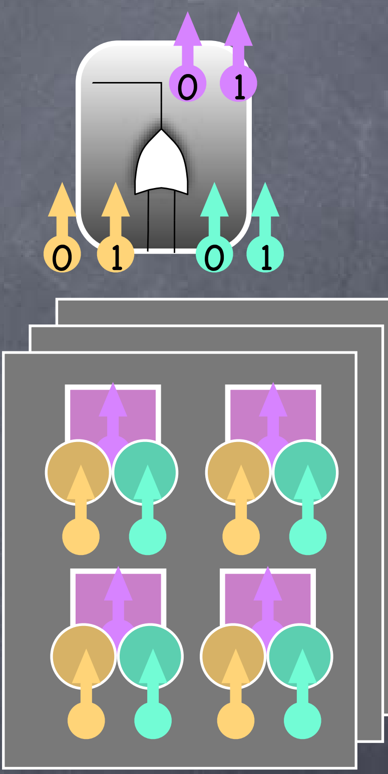


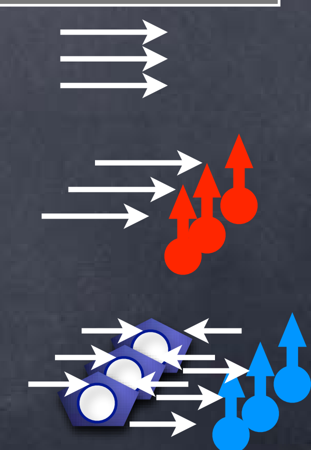



Garbled Circuits

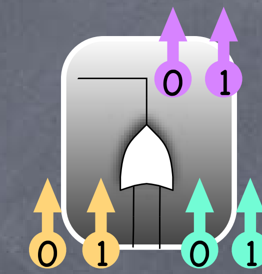
- Recall: Each wire w has two keys ($K_{w=0}$ and $K_{w=1}$). Each garbled gate has 4 boxes with keys for the output wire, locked with keys for input wires
  - Locking: $\mathrm{Enc}_{K_{x=a}}(\mathrm{Enc}_{K_{y=b}}(K_{w=g(a,b)}))$
- An application to MPC: BMR protocol
- Yao's protocol is 1-round, but for only 2 parties
- GMW works for m parties, but is not constant round
- BMR: Use GMW protocol to compute the garbled-circuit based randomized encoding of $f(x_1,\ldots,x_m)$
  - Constant depth encoding ⇒ constant number of rounds.

    Revealing the entire encoding is secure. Decoding (evaluation of GC) done locally by each party.

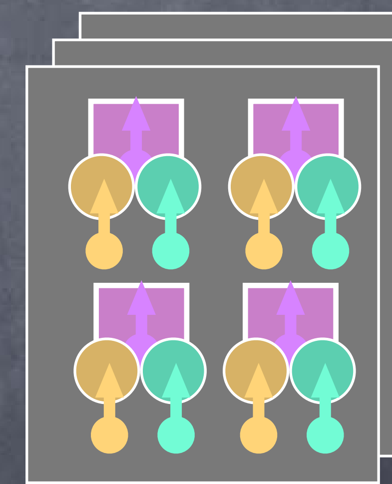




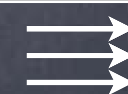
Garbled Circuits



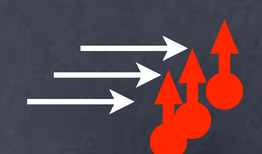
- Recall: Each wire w has two keys ($K_{w=0}$ and $K_{w=1}$). Each garbled gate has 4 boxes with keys for the output wire, locked with keys for input wires
  - Locking: $\mathrm{Enc}_{K_{x=a}}(\mathrm{Enc}_{K_{y=b}}(K_{w=g(a,b)}))$



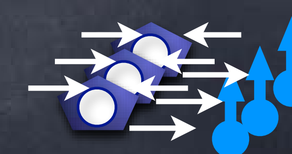
- Information-theoretic garbling: why not just use information-theoretic encryption?
  - One-time pad: $\mathrm{Enc}_K(m) = m \oplus K$



  - But $K_{x=a}$ used to encrypt two values in a gate, $\mathrm{Enc}_{K_{y=0}}(K_{w=g(a,0)})$ and $\mathrm{Enc}_{K_{y=1}}(K_{w=g(a,1)})$



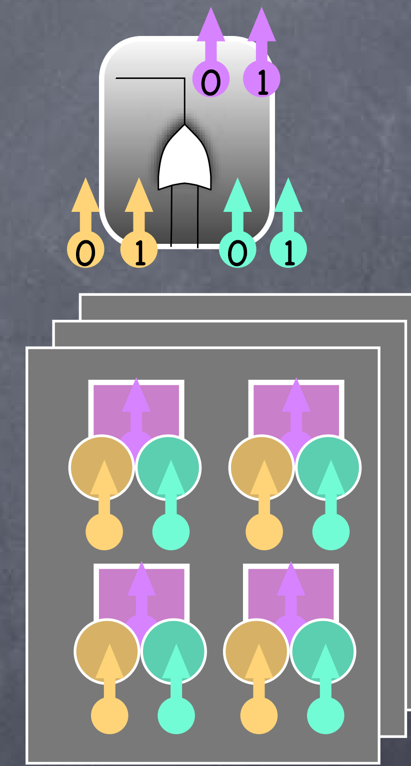
  - If the wire x fans out to t gates, encrypts 2t values
  - Can we still use a one-time pad?

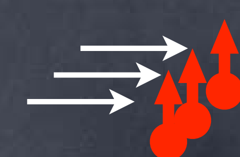


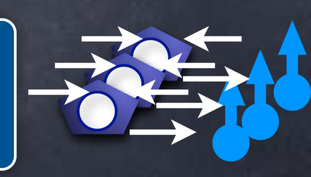
Information-Theoretic Garbled Circuits

- Recall: Each wire w has two keys ($K_{w=0}$ and $K_{w=1}$). Each garbled gate has 4 boxes with keys for the output wire, locked with keys for input wires
    - Locking: $Enc_{K_{x=a}}(Enc_{K_{y=b}}(K_{w=g(a,b)}))$
- Encrypting 2t messages ≡ encrypting a long message
    - Suppose fan-out bounded by t. Then for wires w_i at depth i, enough to have $|K_{w_i=a}| = 2t\ |K_{w_{i-1}=c}|$
    - Key-size at depth d = $O(\ (2t)^d)$ (with 1-bit key at the output)
- Polynomial sized if d is logarithmic and t constant
- Information-theoretic garbled circuits possible for shallow circuits (NC^1)

Alternate constructions avoid bound on t







Recall

Gentry-Sahai-Waters

- Supports messages $\mu \in \{0,1\}$ and NAND operations up to an a priori bounded depth of NANDs
- Public key $M \in \mathbb{Z}_q^{m\times n}$ and private key $\underline{\mathbf{z}}$ s.t. $\underline{\mathbf{z}}^T M$ has small entries
- $\mathrm{Enc}(\mu) = M^T R + \mu G$ where $R \leftarrow \{0,1\}^{m\times km}$ (and $G \in \mathbb{Z}_q^{n\times km}$ the matrix to reverse bit-decomposition)
- $\mathrm{Dec}_z(C)$: $\underline{\mathbf{z}}^T C = \underline{\delta}^T + \mu \underline{\mathbf{z}}^T G$ where $\underline{\delta}^T = e^T R$
- $\mathrm{NAND}(C_1,C_2)$: $G - C_1 \cdot B(C_2)$ (G is a (non-random) encryption of 1)
  - $\underline{\mathbf{z}}^T C_1 \cdot B(C_2) = \underline{\mathbf{z}}^T C_1 \cdot B(C_2) = (\underline{\delta_1}^T + \mu_1 \underline{\mathbf{z}}^T G)\ B(C_2)$

    $= \underline{\delta_1}^T B(C_2) + \mu_1 \underline{\mathbf{z}}^T C_2 = \underline{\delta}^T + \mu_1\mu_2 \underline{\mathbf{z}}^T G$

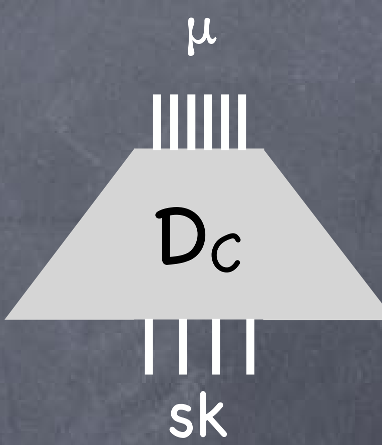
    where $\underline{\delta}^T = \underline{\delta_1}^T B(C_2) + \mu_1 \underline{\delta_2}^T$ has small entries
  - In general, error gets multiplied by km. Allows depth $\approx \log_{km} q$

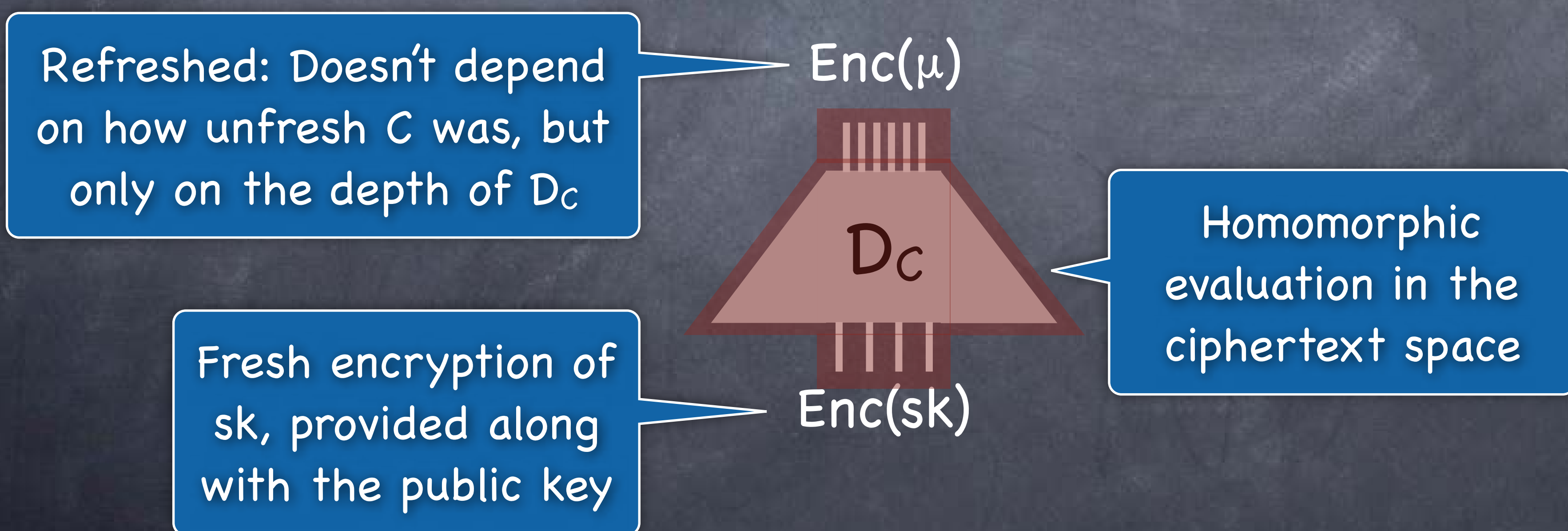
Only "left depth" counts, since $\underline{\delta} \leq k \cdot m \cdot \underline{\delta_1} + \underline{\delta_2}$

Recall

Bootstrapping

- To refresh a given ciphertext C. Also given an encryption of sk (in the public-key). Let D_C be s.t. $D_C(sk) := Dec(C,sk)$.
- Refresh(C,Enc(sk)) = HomomEval(D_C, Enc(sk))
- Need depth of D_C to be strictly less than the depth allowed by the homomorphic encryption scheme





Discussion

That's All Folks!