Advanced Tools from Modern Cryptography

Lecture 16

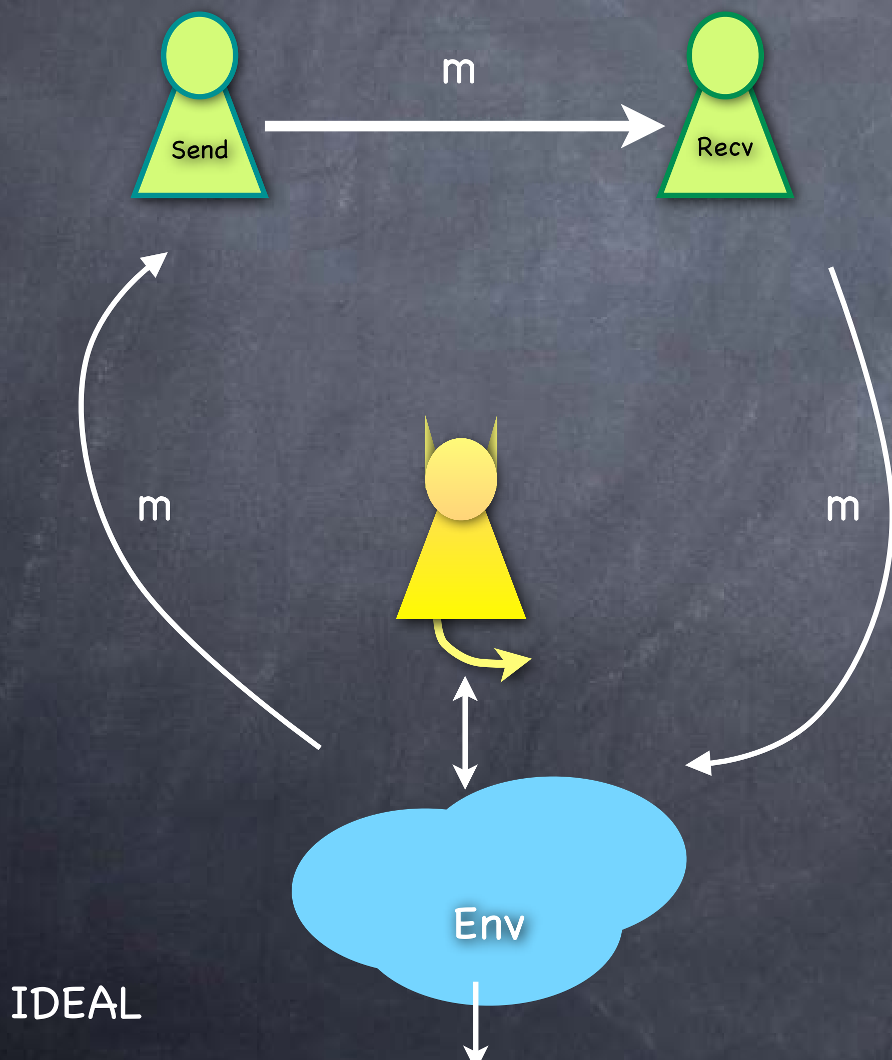
Encryption & Homomorphic Encryption

Public-Key Encryption

a.k.a. asymmetric-key encryption

- Syntax
  - KeyGen outputs (PK,SK) $\leftarrow \mathcal{PK} \times \mathcal{SK}$
  - Enc: $\mathcal{M} \times \mathcal{PK} \times \mathcal{R} \rightarrow \mathcal{C}$
  - Dec: $\mathcal{C} \times \mathcal{SK} \rightarrow \mathcal{M}$
- Correctness
  - $\forall$(PK,SK) $\in$ Range(KeyGen), Dec(Enc(m,PK), SK) = m
- Security
  - Against Chosen-Plaintext Attack: IND-CPA security
  - (Stronger notions of security exist: e.g., IND-CCA security)

SIM-CPA



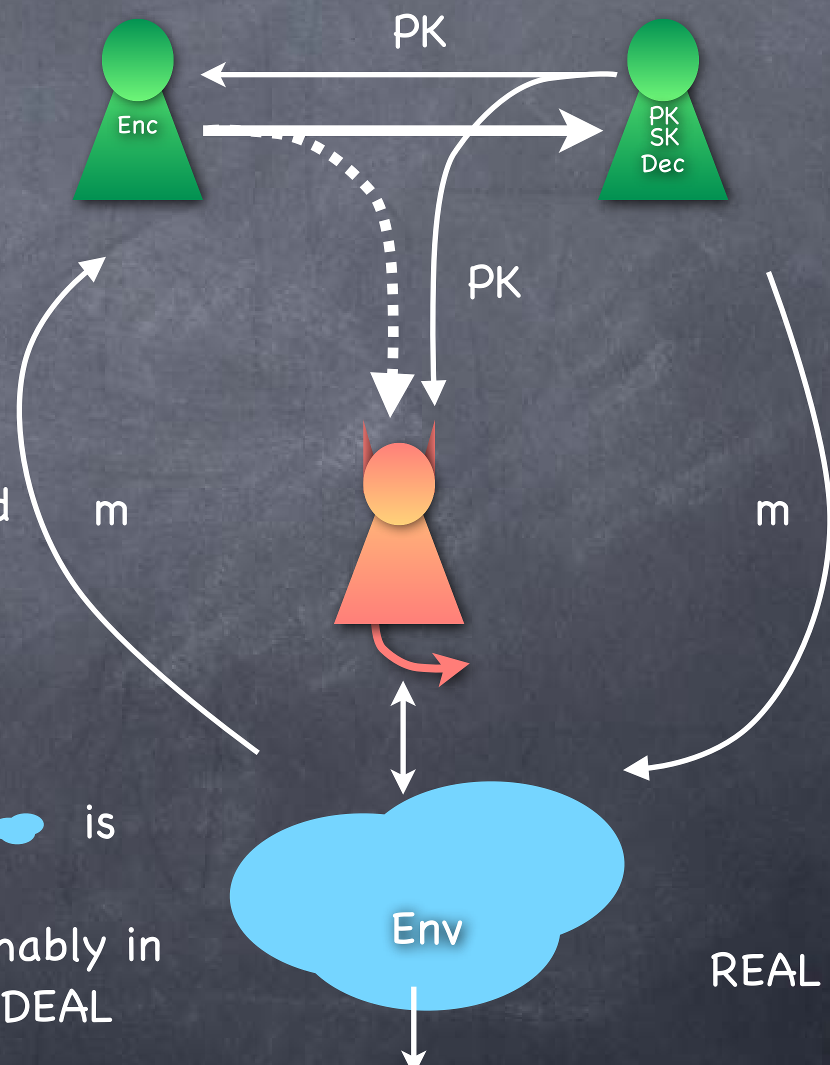
Secure (and correct) if:

∀

∃ s.t.

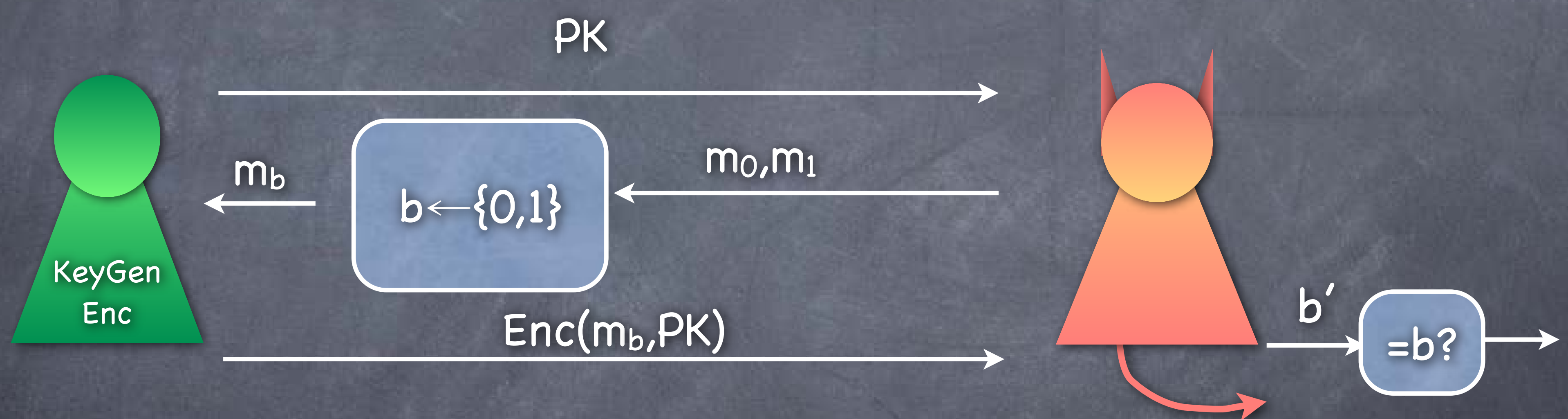
∀

output of is distributed indistinguishably in REAL and IDEAL



IND-CPA Secure PKE

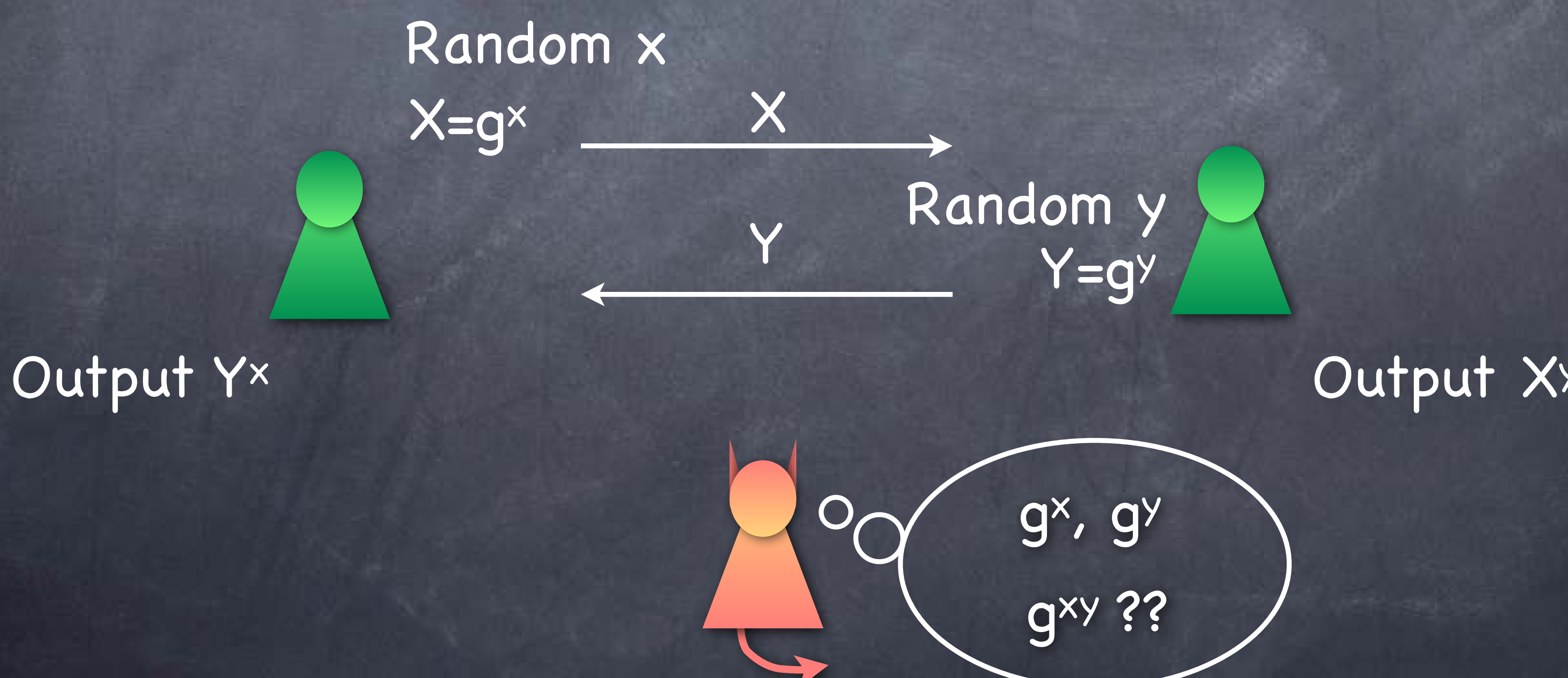
IND-CPA + ~correctness equivalent to SIM-CPA



- IND-CPA secure if for all PPT adversaries $\Pr[b'=b] - 1/2 \leq \nu(k)$

Diffie-Hellman Key-exchange

- A candidate for how Alice and Bob could generate a shared key, which is "hidden" from Eve



Why DH-Key-exchange could be secure

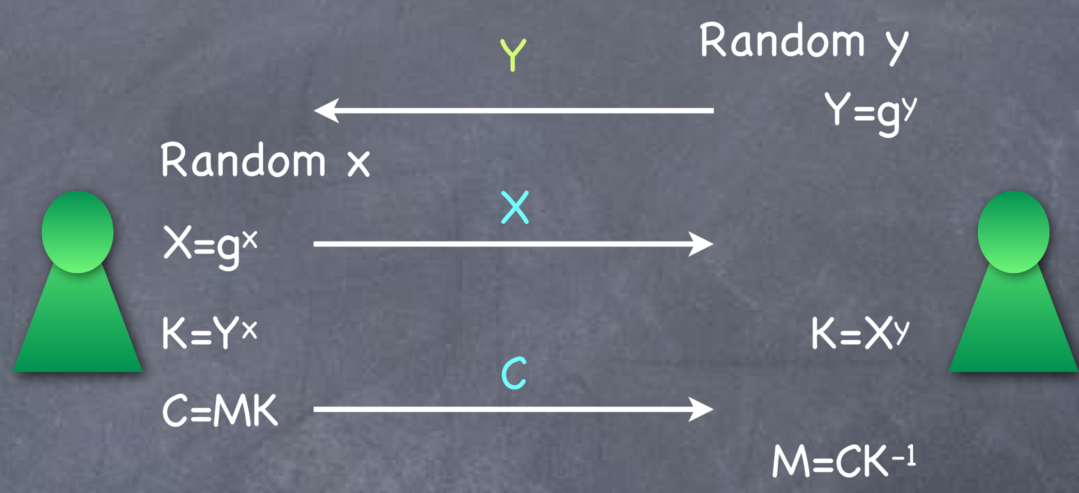
- Given g^x, g^y for random x, y, g^{xy} should be “hidden”
  - i.e., could still be used as a pseudorandom element
  - i.e., $(g^x, g^y, g^{xy}) \approx (g^x, g^y, R)$
- [Recall] Decisional DH Assumption: A family of cyclic groups, with

  $$\{(g^x, g^y, g^{xy})\}_{(G,g)\leftarrow \textbf{GroupGen};\ x,y\leftarrow[|G|]} \approx \{(g^x, g^y, g^r)\}_{(G,g)\leftarrow \textbf{GroupGen};\ x,y,r\leftarrow[|G|]}$$

  where (G,g) s.t. g is generator for G (and typically |G| prime, so that operations in exponent are in a field)
- There are families of number-theoretic and algebraic (elliptic curve) groups for which DDH is assumed to hold

El Gamal Encryption

- Based on DH key-exchange
- Bob's "message" in the key-exchange is his PK
- Alice's message in the key-exchange and the message masked with this key together form a single ciphertext



KeyGen: PK=(G,g,Y), SK=(G,g,y)

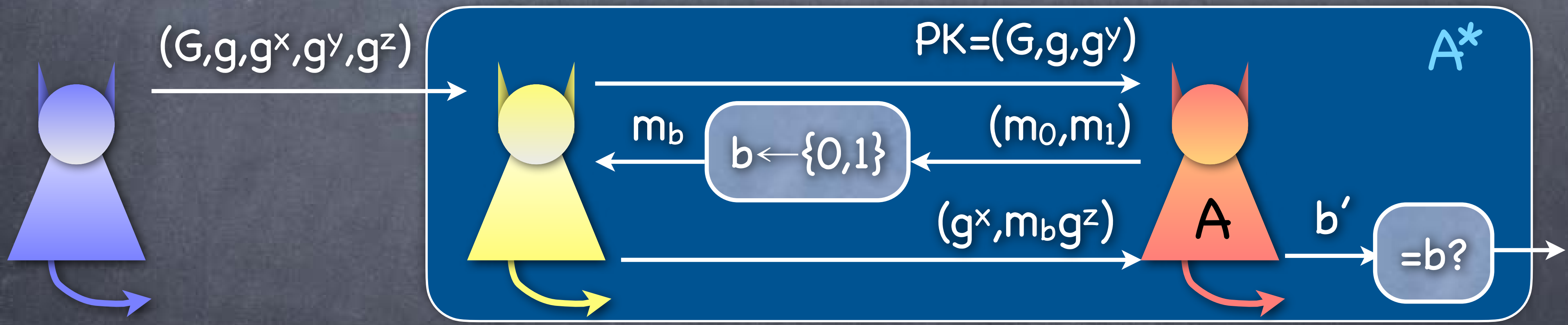
$\mathrm{Enc}_{(G,g,Y)}(M) = (X=g^x, C=MY^x)$

$\mathrm{Dec}_{(G,g,y)}(X,C) = CX^{-y}$

- KeyGen uses GroupGen to get (G,g)
- x, y uniform from [|G|]
- Message encoded into group element, and decoded

Security of El Gamal

- El Gamal IND-CPA secure if DDH holds (for the collection of groups used)
  - Construct a DDH adversary A* given an IND-CPA adversary A



  - When z=xy, exactly IND-CPA experiment:
    A* outputs 1 with probability = 1/2 + advantage of A.
  - When z=random, A* outputs 1 with probability = 1/2

Homomorphic Encryption

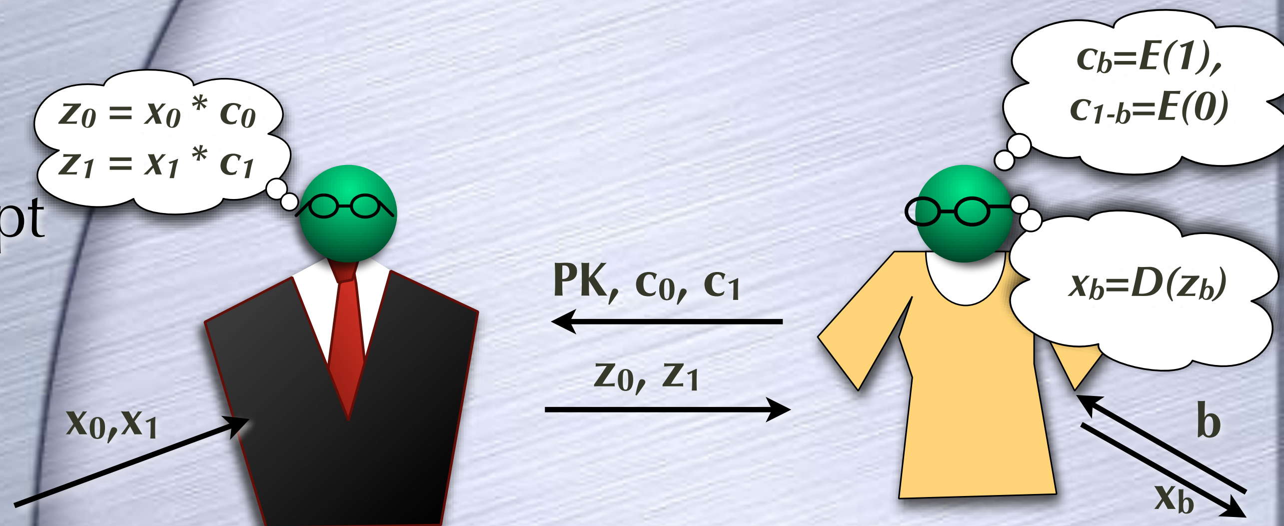
- Group Homomorphism: Two groups G and G′ are homomorphic if there exists a function (homomorphism) $f:G \rightarrow G'$ such that for all $x,y \in G$, $f(x) +_{G'} f(y) = f(x +_G y)$
- Homomorphic Encryption: A CPA secure (public-key) encryption s.t. $\mathrm{Dec}(C) +_M \mathrm{Dec}(D) = \mathrm{Dec}(C +_C D)$ for ciphertexts C, D
  - i.e. $\mathrm{Enc}(x) +_C \mathrm{Enc}(y)$ is like $\mathrm{Enc}(x +_M y)$
  - Interesting when $+_C$ doesn't require the decryption key
- e.g. El Gamal: $(g^{x_1}, m_1 Y^{x_1}) \times (g^{x_2}, m_2 Y^{x_2}) = (g^{x_3}, m_1 m_2 Y^{x_3})$

Rerandomization

- Often (but not always) another property is required of a homomorphic encryption scheme
- **Unlinkability**
  - For any two ciphertexts c_x=Enc(x) and c_y=Enc(y), Add(c_x,c_y) should be identically distributed as Enc(x $+_M$ y). Add is a randomized operation
- Alternately, a **ReRand** operation s.t. for all valid ciphertexts c_x, ReRand(c_x) is identically distributed as Enc(x)
  - Then, we can let Add(c_x,c_y) = ReRand(c_x $+_c$ c_y) where $+_c$ may be deterministic
  - Rerandomization useful even without homomorphism
- e.g. El Gamal: Rerand maps $(g^x, mY^x) \mapsto (g^x g^r, mY^x Y^r)$ for $r \leftarrow [|G|]$

An OT Protocol (for passive corruption)

- Using an (unlinkable) rerandomizable encryption scheme
  - Receiver picks (PK,SK). Sends PK and $c_b = E(1)$, $c_{1-b} = E(0)$,
  - Sender "multiplies" c_i with x_i: $1{*}c := \text{ReRand}(c)$, $0{*}c := E(0)$
- Simulation for passive-corrupt receiver: set $z_b = E(x_b)$ and $z_{1-b} = E(0)$
- Simulation for passive-corrupt sender: let c_0, c_1 be $E(1)$, say
  - In both cases, send input from environment to functionality



Homomorphic Encryption for MPC

- Recall GMW (passive-secure): each input was secret-shared among the parties, and computed on shares, using pair-wise OTs for $\times$ gates
- Alternate approach that avoids pair-wise communication: each wire value is kept encrypted, publicly, and the key is kept shared
  - All parties encrypt their inputs and publish
    all communication will be of this form
  - Evaluate each wire using homomorphism (coming up)
  - Finally decrypt the output wire value using threshold decryption
    - Threshold decryption: KeyGen protocol so that PK is public and SK shared; Decryption protocol that lets the parties decrypt a ciphertext keeping their SK shares private

Threshold El Gamal (Passive Security)

- Goal: n parties to generate a PK for El Gamal, so that SK is shared amongst them. Can decrypt messages only if all n parties come together. Will require security against passive corruption.
- Distributed Key-Generation:
  - $(G,g) \leftarrow$ Groupgen by Party$_1$ (DDH should hold for Party$_1$ too)
  - Each Party$_i$ picks random exponent y_i and publishes $Y_i = g^{y_i}$
  - All parties compute $Y = \Pi_i Y_i$. Public-key = (G,g,Y)
  - Secret-key = (G,g,y), where $y := \Sigma_i y_i$ (secret). Note: $Y = g^y$
- Encryption as in El Gamal
- Distributed Decryption: Given ciphertext (X,C), each party publishes $K_i^{-1} = X^{-y_i}$. All parties compute $K^{-1} = \Pi_i K_i^{-1}$ and $M = CK^{-1}$

Homomorphic Encryption for MPC

- Passive-securely computing using homomorphism
  - Notation: Encrypted values shown as [m] etc.
  - Operations available: [x]+[y] = [x+y], and a*[x] = [ax]
    - Also, distributed key generation and threshold decryption
- Addition directly, without communication
- Multiplication: All parties have [x] and [y]. Need [xy].
  - Each party P_i picks a_i,b_i and publishes [a_i], [b_i], [$a_i y$], [$b_i x$]
  - All compute [x+a], [y+b], [ay], [bx] where $a = \Sigma_i a_i$ and $b = \Sigma_i b_i$
  - Each P_i publishes [$a_i b$] = a_i*[b], and all compute [ab]
  - Threshold decrypt (x+a),(y+b). Compute [z] where z=(x+a)(y+b).
  - All compute [xy] = [z] - [ay] - [bx] - [ab]

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