Some Project Ideas

- Read & Write something
  - Constructions not covered in class (e.g., McEliece PKE, lattice-based PKE), primitives not covered (e.g., Zero-Knowledge, Oblivious Transfer), proofs not covered (e.g., security of TLS),...
- Implementation project
  - Make something
    - Slow and secure crypto (e.g., SKE and/or Digital Signatures from OWP, full-domain CRHF from DL,...)
    - Higher-level applications (e.g., "simple-TLS", Off-the-record messaging, things you can do with a block-cipher...)
    - A library with a cleaner API for encryption/authentication
  - Break something
    - e.g., use a constraint-solver to break (broken) block-ciphers

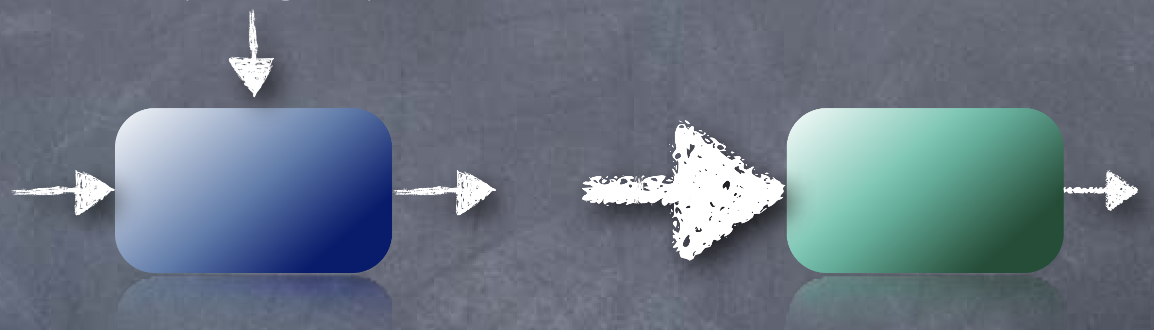
Hash Functions

Lecture 14

Flavours of collision resistance

A Tale of Two Boxes

- The bulk of today's applied cryptography works with two magic boxes
  - Block Ciphers
  - Hash Functions



- Block Ciphers: Best modeled as (strong) Pseudorandom Permutations, with inversion trapdoors
  - Often more than needed (e.g. SKE needs only PRF)
- Hash Functions:
  - Some times modeled as Random Oracles!
    - Schemes relying on this can often be broken
  - Today: understanding security requirements on hash functions

Hash Functions

- "Randomized" mapping of inputs to shorter hash-values
- Hash functions are useful in various places
  - In data-structures: for efficiency
    - Intuition: hashing removes worst-case effects
  - In cryptography: for "integrity"
- Primary use: Domain extension (compress long inputs, and feed them into boxes that can take only short inputs)
  - Typical security requirement: "collision resistance"
  - Also sometimes: some kind of unpredictability

Hash Function Family

- Hash function $h:\{0,1\}^{n(k)} \to \{0,1\}^{t(k)}$
  - Compresses
- A family
  - Alternately, takes two inputs, the index of the member of the family, and the real input
- Efficient sampling and evaluation
- Idea: when the hash function is randomly chosen, "behaves randomly"
  - Main goal: to "avoid collisions". Will see several variants of the problem

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$		$h_N(x)$
000	0	0	0	1	...	1
001	0	0	1	1		1
010	0	1	0	1		1
011	0	1	1	0		1
100	1	0	0	1		1
101	1	0	1	0		1
110	1	1	0	1		1
111	1	1	1	0		1

Hash Functions in Crypto Practice

- A single fixed function
  - e.g. SHA-3, SHA-256, SHA-1, MD5, MD4
  - Not a family ("unkeyed")
  - (And no security parameter knob)
- Not collision-resistant under any of the following definitions
- Alternately, could be considered as have already been randomly chosen from a family (and security parameter fixed too)
  - Usually involves hand-picked values (e.g. "I.V." or "round constants") built into the standard

Degrees of Collision-Resistance

- If for all PPT A, Pr[$x \neq y$ and $h(x)=h(y)$] is negligible in the following experiment:
  - $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$: Combinatorial Hash Functions (even non-PPT A)
  - $A \rightarrow x$; $h \leftarrow \mathcal{H}$; $A(h) \rightarrow y$: Universal One-Way Hash Functions
  - $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$: Collision-Resistant Hash Functions
- Also useful sometimes: A gets only oracle access to h(.) (weak). Or, A gets any coins used for sampling h (strong).
- CRHF the strongest; UOWHF still powerful (will be enough for digital signatures)

Degrees of Collision-Resistance

- Weaker variants of CRHF/UOWHF (where x is random)
  - $h \leftarrow \mathcal{H}$; $x \leftarrow X$; $A(h,h(x)) \rightarrow y$ ($y=x$ allowed)
    - Pre-image collision resistance if $h(x)=h(y)$ w.n.p
    - i.e., $f(h,x) := (h,h(x))$ is a OWF (and h compresses)
  - $h \leftarrow \mathcal{H}$; $x \leftarrow X$; $A(h,x) \rightarrow y$ ($y \neq x$)
    - Second Pre-image collision resistance if $h(x)=h(y)$ w.n.p
  - Incomparable (neither implies the other) [Exercise]
- CRHF implies second pre-image collision resistance and, if compressing, then pre-image collision resistance [Exercise]

A.k.a One-Way Hash Function

Hash Length

- If range of the hash function is too small, not collision-resistant
  - If range poly-size (i.e. hash log-long), then non-negligible probability that two random x, y provide collision
- In practice interested in minimizing the hash length (for efficiency)
  - Generic collision-finding attack: birthday attack
    - Look for a collision in a set of random hashes (needs only oracle access to the hash function)
      - Expected size of the set before collision: $O(\sqrt{|range|})$
  - Birthday attack effectively halves the hash length (say security parameter) over "naïve attack"

Universal Hashing

- Combinatorial HF: $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$. $h(x)=h(y)$ w.n.p
- Even better: 2-Universal Hash Functions
  - "Uniform" and "Pairwise-independent"
  - $\forall x,z$ $\Pr_{h \leftarrow \mathcal{H}}[\ h(x)=z\ ] = 1/|Z|$ (where $h:X \rightarrow Z$)
  - $\forall x \neq y,w,z$ $\Pr_{h \leftarrow \mathcal{H}}[\ h(x)=w,\ h(y)=z\ ] = 1/|Z|^2$
    - $\Rightarrow$ $\forall x \neq y$ $\Pr_{h \leftarrow \mathcal{H}}[\ h(x)=h(y)\ ] = 1/|Z|$

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$
0	0	0	1	1
1	0	1	0	1
2	1	0	0	1

Negligible collision-probability if super-polynomial-sized range

- k-Universal:
  - $\forall x_1..x_k$ (distinct), $z_1..z_k$, $\Pr_{h \leftarrow \mathcal{H}}[\forall i\ h(x_i)=z_i\ ] = 1/|Z|^k$
- Inefficient example: $\mathcal{H}$ set of all functions from X to Z
  - But we will need all $h \in \mathcal{H}$ to be succinctly described and efficiently evaluable

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Negligible collision-probability if super-polynomial-sized range

- e.g. $h_{a,b}(x) = ax+b$ (in a finite field, $X=Z$)
  - $\Pr_{a,b}[\ ax+b = z\ ] = \Pr_{a,b}[\ b = z-ax\ ] = 1/|Z|$
  - $\Pr_{a,b}[\ ax+b = w,\ ay+b = z] = ?$ Exactly one (a,b) satisfying the two equations (for $x \neq y$)
    - $\Pr_{a,b}[\ ax+b = w,\ ay+b = z] = 1/|Z|^2$
  - But does not compress!

Universal Hashing

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- Even better: 2-Universal Hash Functions
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  - $\forall x,z$ $Pr_{h \leftarrow \mathcal{H}}$ [$h(x)=z$] = $1/|Z|$ (where $h:X \rightarrow Z$)
  - $\forall x \neq y,w,z$ $Pr_{h \leftarrow \mathcal{H}}$ [$h(x)=w$, $h(y)=z$] = $1/|Z|^2$
    - $\Rightarrow$ $\forall x \neq y$ $Pr_{h \leftarrow \mathcal{H}}$ [$h(x)=h(y)$] = $1/|Z|$

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Negligible collision-probability if super-polynomial-sized range

- e.g. $h'_h(x)$ = Chop($h(x)$) where h from a (possibly non-compressing) 2-universal HF
  - Chop a t-to-1 map from Z to Z' (e.g. removes last bit: 2-to-1)
    - Pr_h [Chop($h(x)$) = w, Chop($h(y)$) = z]
      = Pr_h [$h(x)$ = w0 or w1, $h(y)$ = z0 or z1] = $4/|Z|^2 = 1/|Z'|^2$

UOWHF

- Universal One-Way HF: A→x; h←$\mathcal{H}$; A(h)→y. h(x)=h(y) w.n.p
- Can be constructed from OWF
- Much easier to see: OWP ⇒ UOWHF
  - $F_h(x) = h(f(x))$, where f is a OWP and h from a UHF family
    - s.t. h compresses by a bit (i.e., 2-to-1 maps), and
    - for all z, z', w, can solve for h s.t. h(z) = h(z') = w
  - Is a UOWHF [Why?]
    BreakOWP(z) { get x ← A; sample random w; give A h s.t. h(z)=h(f(x))=w; if A→y s.t. h(f(y))=w, output y; }
  - Gives a UOWHF that compresses by 1 bit (same as the UHF)
    - Will see later, how to extend the domain to arbitrarily long strings (without increasing output size)