Hash Functions in Action

Lecture 15

RECALL

Hash Functions

- Main syntactic feature: Variable input length to fixed length output
- Primary requirement: collision-resistance
  - If for all PPT A, Pr[x≠y and h(x)=h(y)] is negligible in the following experiment:
    - $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$: Combinatorial Hash Functions
    - $A \rightarrow x$; $h \leftarrow \mathcal{H}$; $A(h) \rightarrow y$: Universal One-Way Hash Functions
    - $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$: Collision-Resistant Hash Functions
      - $h \leftarrow \mathcal{H}$; $A^h \rightarrow (x,y)$: Weak Collision-Resistant Hash Functions

Typically used

- Also often required: "unpredictability"
- Already saw: a 2-UHF (chop(ax+b)) and UOWHF
- Today: CRHF constructions. Domain Extension. Applications of hash functions

RECALL

UOWHF

- Universal One-Way HF: A→x; h←$\mathcal{H}$; A(h)→y. h(x)=h(y) w.n.p
- Can be constructed from OWF
- Much easier to see: OWP ⇒ UOWHF
  - $F_h(x)$ = h(f(x)), where f is a OWP and h from a UHF family
    - s.t. h compresses by a bit (i.e., 2-to-1 maps), and
    - for all z, z', w, can solve for h s.t. h(z) = h(z') = w
  - Is a UOWHF [Why?]

    BreakOWP(z) { get x ← A; sample random w; give A h s.t. h(z)=h(f(x))=w; if A→y s.t. h(f(y))=w, output y; }
  - Gives a UOWHF that compresses by 1 bit (same as the UHF)
    - Will see later, how to extend the domain to arbitrarily long strings (without increasing output size)

UOWHF

If not unique, uniformly sample a solution for h

- $F_h(x) = h(f(x))$, where f is a OWP and h from a UHF family
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BreakOWP(z) { get x ← A; sample random w; give A h s.t. h(z)=h(f(x))=w; if A→y s.t. h(f(y))=w, output y; }

  - Idea: force UOWHF adversary to invert f
  - Set up h so that $F_h(x) = h(z)$. Only collision, i.e., $y \neq x$ s.t. $F_h(x) = F_h(y)$ is $y=f^{-1}(z)$
  - BreakOWP is efficient as h can be efficiently solved ✓
  - BreakOWP has same advantage as A has against UOWHF? Yes, if h is uniform (independent of x)
    - Holds because z, w picked uniformly ✓

CRHF

- Collision-Resistant HF: $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$. $h(x)=h(y)$ w.n.p
- Not known to be possible from OWF/OWP alone
  - "Impossibility" (blackbox-separation) known
- Possible from "claw-free pair of permutations"
  - In turn from hardness of discrete-log, factoring, and from lattice-based assumptions
- Also from "homomorphic one-way permutations", and from homomorphic encryptions
  - All candidates use mathematical operations that are considered computationally expensive

CRHF

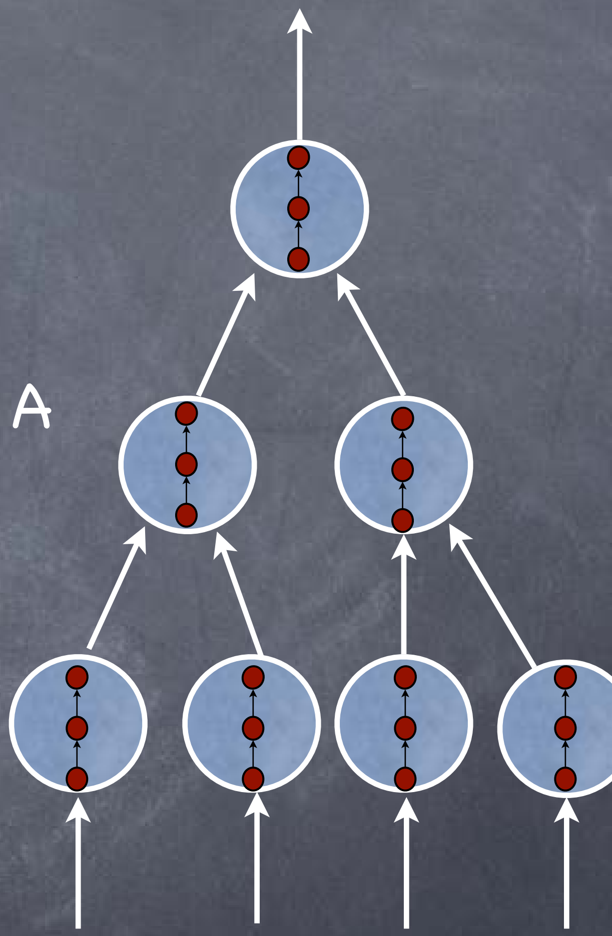
- CRHF from discrete log assumption:
  - Suppose $\mathbb{G}$ a group of prime order q, where DL is considered hard (e.g. $\mathbb{QR}_p^*$ for p=2q+1 a safe prime)
  - $h_{g1,g2}(x_1,x_2) = g_1^{x1} g_2^{x2}$ (in $\mathbb{G}$) where g_1, $g_2 \neq 1$ (hence generators)
  - A collision: $(x_1,x_2) \neq (y_1,y_2)$ s.t. $h_{g1,g2}(x_1,x_2) = h_{g1,g2}(y_1,y_2)$
    - Then $(x_1,x_2) \neq (y_1,y_2) \Rightarrow x_1 \neq y_1$ and $x_2 \neq y_2$ [Why?]
    - Then $g_2 = g_1^{(x1-y1)/(x2-y2)}$ (exponents in $\mathbb{Z}_q^*$)
      - i.e., for some base g_1, can compute DL of g_2 (a random non-unit element). Breaks DL!
  - Hash halves the size of the input

Domain Extension

- Full-domain hash: hash arbitrarily long strings to a single hash value
  - So far, UOWHF/CRHF which have a fixed domain
- First, simpler goal: a extend to a larger, fixed domain
  - Assume we are given a hash function from two blocks to one block (a block being, say, k bits)
    - What if we can compress by only one bit (e.g., our UOWHF construction)?
      - Can just apply repeatedly to compress by t bits

Domain Extension

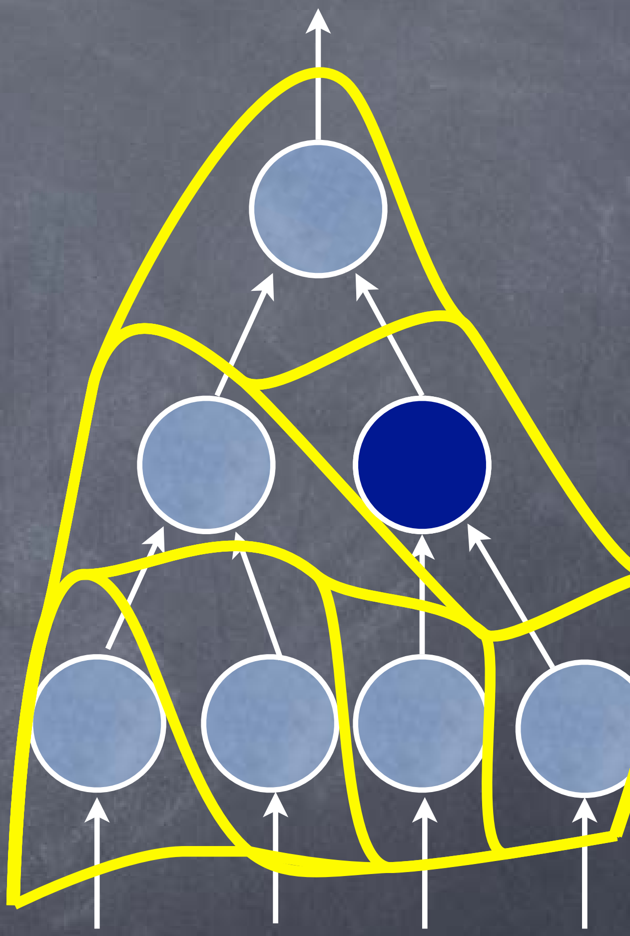
- Given an compose hash functions more efficiently, using a "Merkle tree"
  - Suppose basic hash **from $\{0,1\}^{2k}$ to $\{0,1\}^k$**. A hash function from $\{0,1\}^{8k}$ to $\{0,1\}^k$ using a tree of depth 3
    - If basic hash from $\{0,1\}^{2k}$ to $\{0,1\}^{2k-1}$, first construct new basic hash from $\{0,1\}^{2k}$ to $\{0,1\}^k$, by repeated hashing
  - Any tree can be used, with consistent I/O sizes



- Independent hashes or same hash?
  - Depends!

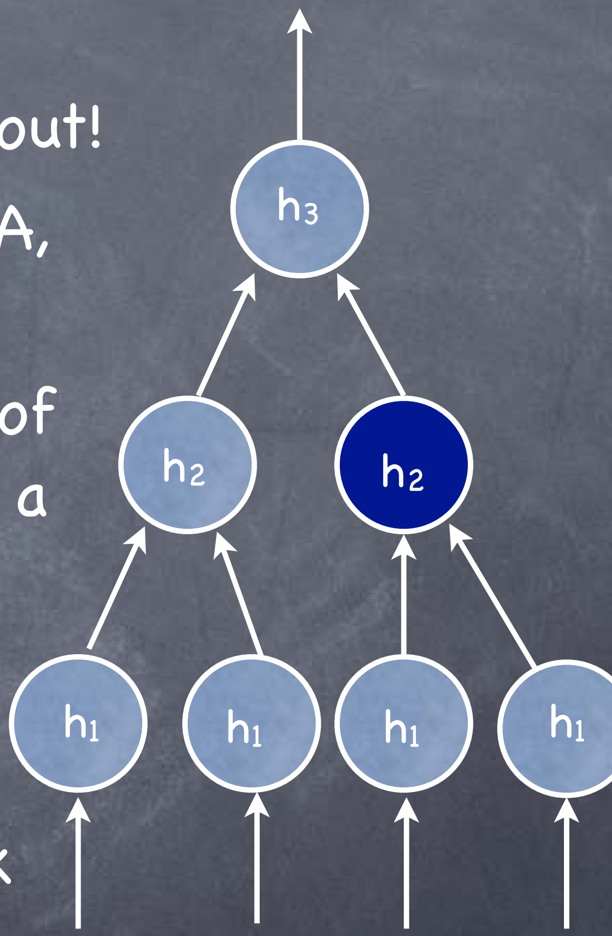
Domain Extension for CRHF

- For CRHF, same basic hash used through out the Merkle tree. Hash description same as for a single basic hash
- If a collision ($(x_1 \ldots x_n)$, $(y_1 \ldots y_n)$) over all, then some collision (x',y') for basic hash
  - Consider moving a "frontline" from bottom to top
    - Collision at some step (different values on i^{th} front, same on $i+1^{st}$); gives a collision for basic hash
- $A^*(h)$: run $A(h)$ to get $(x_1 \ldots x_n)$, $(y_1 \ldots y_n)$. Move frontline to find (x',y')



Domain Extension for UOWHF

- For UOWHF, can't use same basic hash throughout!
- A* has to output an x′ on getting ($x_1 \ldots x_n$) from A, before getting h
  - Can guess a random node (i.e., random pair of frontlines) where collision occurs, but if not a leaf, can't compute x′ until h is fixed!
- Solution: a different h for each level of the tree (i.e., no ancestor/successor has same h)
  - To compute x′: Get ($x_1 \ldots x_n$) from A. Then pick a random node (say at level i), pick h_j for levels below i, and compute input to the node; let this be x′.
  - On getting h, plug it in as h_i, pick h_j for remaining levels; give h's to A and get ($y_1 \ldots y_n$); compute y′ and output it.

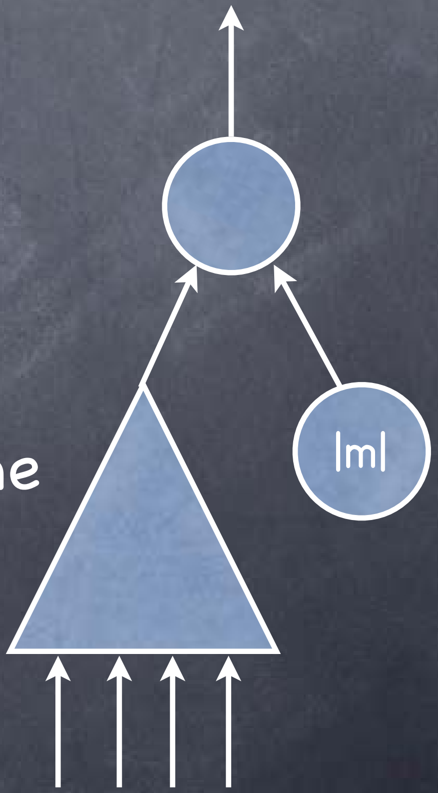


UOWHF vs. CRHF

- UOWHF has a weaker guarantee than CRHF
- UOWHF can be built based on OWF (we saw based on OWP), where as CRHF "needs stronger assumptions"
  - But "usual" OWF candidates suffice for CRHF too (we saw construction based on discrete-log)
- Domain extension of CRHF is simpler, with no blow-up in the description size. For UOWHF description increases logarithmically in the input size
- UOWHF theoretically important (based on simpler assumptions, good if paranoid), but CRHF can substitute for it
- Current practice: much less paranoid; faith on efficient, ad hoc (and unkeyed) constructions (though increasingly under attack)

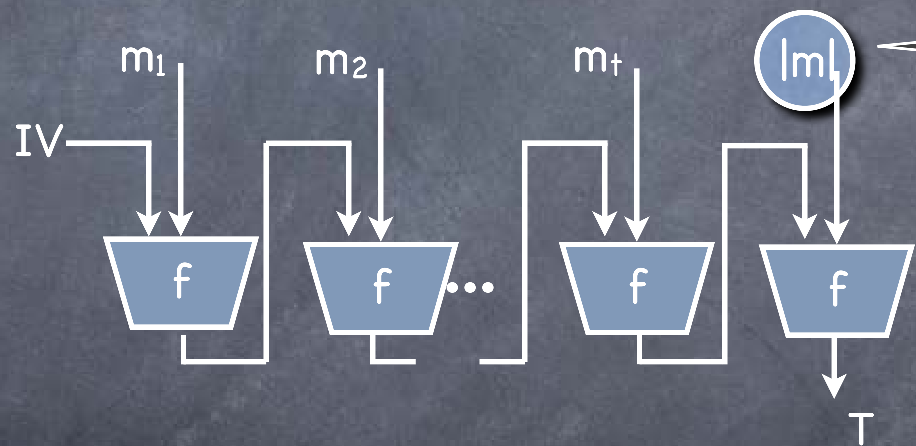
Domain Extension

- Full-domain hash: hash arbitrarily long strings to a single hash value
  - Merkle-Tree construction extends the domain to any fixed input length
- Hash the message length (number of blocks) along with the original hash
  - Collision in the new hash function gives either collision at the top level, or if not, collision in the original Merkle tree and for the same message length



Hash Functions in Practice

- A single function, not a family (e.g. SHA-3, SHA-256, MD4, MD5)
- Often from a fixed input-length compression function
- Merkle-Damgård iterated hash function, MD^f:



Collision resistance even with variable input-length.

Note: Unlike MACs, here "length-extension" is OK, as long as it results in a different hash value

- If f collision resistant (not as "keyed" hash, but "concretely"), then so is MD^f (for any IV)
  - If f modelled as a Random Oracle, MD^f is a "public-use RO." If f modelled as an "Ideal Cipher," MD^f is "plaint-text aware."