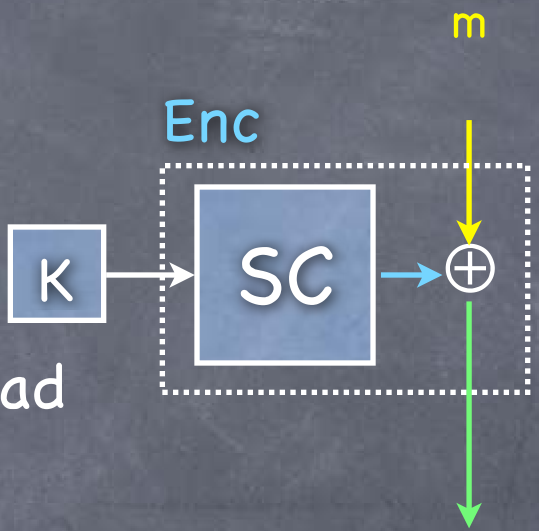
Symmetric-Key Encryption: One-Way Functions

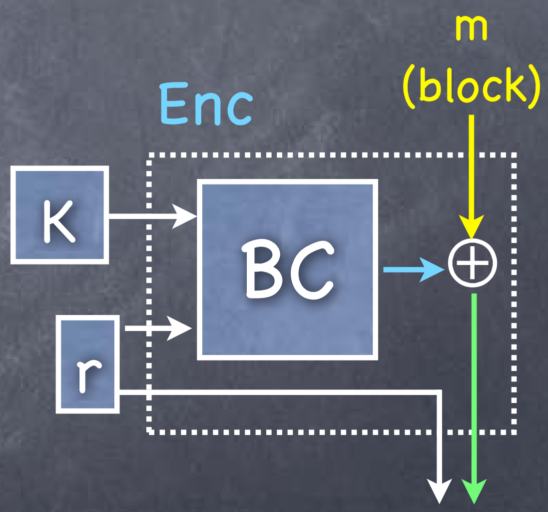
Lecture 6

PRG from One-Way Permutations

Story So far



- PRG (i.e., a Stream Cipher) for one-time SKE
  - "Mode of operation": msg ⊕ pseudorandom pad
- PRF (i.e., a Block Cipher) for full-fledged SKE
  - Many standard modes of operation: OFB, CTR, CBC, ...
  - All provably CPA-secure if the Block Cipher is a PRF (or PRP with trapdoor, for CBC). CTR mode is recommended (most efficient)

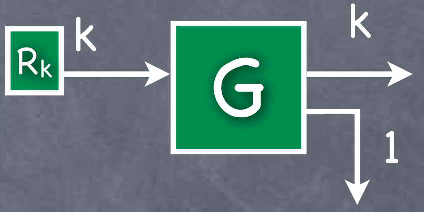


- In practice, fast/complex constructions for Block Ciphers
  - But in principle, a PRF can be securely built from a PRG

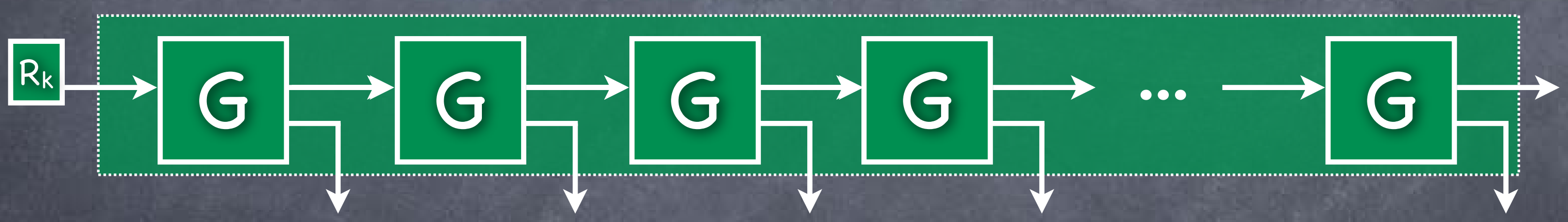
RECALL

PRG

coming up



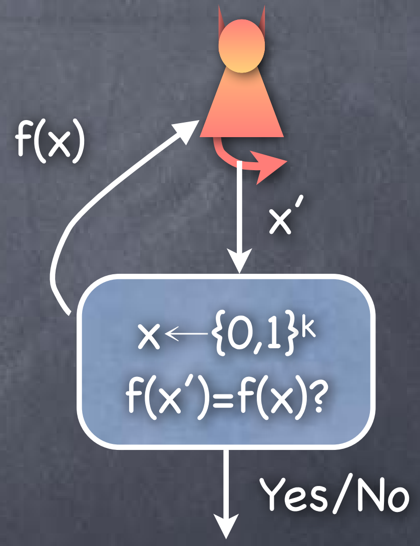
- Can build a PRG from a one-bit stretch PRG, G_k: $\{0,1\}^k \rightarrow \{0,1\}^{k+1}$
  - Can use part of the PRG output as a new seed



  - Stream cipher: the intermediate seeds are never output, can keep stretching on demand (for any "polynomial length")

One-Way Function

- f_k: $\{0,1\}^k \rightarrow \{0,1\}^{n(k)}$ is a one-way function (OWF) if
  - f is polynomial time computable
  - For all (non-uniform) PPT adversary, probability of success in the "OWF experiment" is negligible
  - Note: x may not be completely hidden by f(x)



One-Way Function Candidates

- Integer factorization:
  - $f_{mult}(x,y) = x \cdot y$
  - Input distribution: (x,y) random k-bit primes
  - Fact: taking input domain to be the set of all k-bit integers, with input distribution being uniform over it, will also work (if k-bit primes distribution works)
    - In that case, it is important that we require $|x|=|y|=k$, not just $|x \cdot y|=2k$ (otherwise, 2 is a valid factor of x.y with 3/4 probability)

One-Way Function Candidates

- Solving Subset Sum:
  - $f_{subsum}(x_1 \ldots x_k, S) = (x_1 \ldots x_k, \sum_{i \in S} x_i)$
  - Input distribution: x_i k-bit integers, $S \subseteq \{1 \ldots k\}$. Uniform
  - Inverting f_{subsum} known to be NP-hard, but assuming that it is a OWF is "stronger" than assuming $P \neq NP$
- Note: $(x_1, \ldots, x_k)$ is "public" (given as part of the output to be inverted)
  - <u>OWF Collection</u>: A collection of subset sum problems, all with the same $(x_1, \ldots, x_k)$ (and independent S)

One-Way Function Candidates

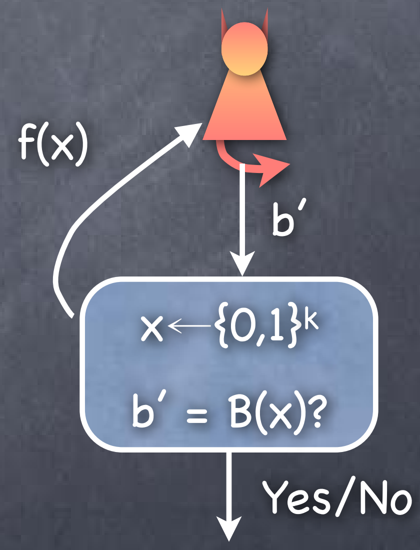
- Goldreich's Candidate:
  - $f_{Goldreich}(x, S_1,\ldots,S_n, P) = (P(x|_{S1}),\ldots,P(x|_{Sn}),S_1,\ldots,S_n, P)$
    - $x \in \{0,1\}^k$, $S_i \subseteq [k]$ with $|S_i|=d$, $P:\{0,1\}^d \rightarrow \{0,1\}$, and $x|_S$ stands for x restricted to indices in S
  - Input distribution: uniformly random with the requisite structure
- OWF Collection: $(S_1,\ldots,S_n,P)$ forms the index

One-Way Function Candidates

- Rabin OWF: $f_{Rabin}(x; n) = (x^2 \bmod n, n)$, where $n = pq$, and p, q are random k-bit primes, and x is uniform from $\{0...n\}$
  - OWF collection: indexed by n
- More: e.g, Discrete Logarithm (uses as index: a group & generator), RSA function (uses as index: $n=pq$ & an exponent e).
  - Later

Hardcore Predicate

- OWFs provide no hiding property that can be readily used
- E.g. every single bit of (random) x may be significantly predictable from f(x), even if f is a OWF [Exercise]
- Hardcore predicate associated with f: a function B such that B(x) remains "completely" hidden given f(x)



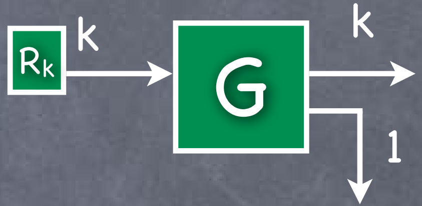
Hardcore Predicates

- For candidate OWFs, often hardcore predicates known
  - e.g. if $f_{Rabin}(x;n)$ is a OWF, then LSB(x) is a hardcore predicate for it
    - **Reduction**: Given an algorithm for finding LSB(x) from $f_{Rabin}(x;n)$ for random x, one can use it (efficiently) to invert f_{Rabin}

Goldreich-Levin Predicate

- Given any OWF f, can slightly modify it to get a OWF g_f such that
  - g_f has a simple hardcore predicate
  - g_f is almost as efficient as f; is a permutation if f is one
- $g_f(x,r) = (f(x), r)$, where $|r|=|x|$
  - Input distribution: x as for f, and r independently random
- GL-predicate: $B(x,r) = \langle x,r \rangle$ (dot product of bit vectors)
  - Can show that a predictor of B(x,r) with non-negligible advantage can be turned into an inversion algorithm for f
    - Predictor for B(x,r) is a "noisy channel" through which x, encoded as $(\langle x,0\rangle,\langle x,1\rangle \ldots \langle x,2^{|x|}-1\rangle)$ (Walsh-Hadamard code), is transmitted. Can efficiently recover x by error-correction (local list decoding).

PRG from One-Way Permutations



- One-bit stretch PRG, G_k: $\{0,1\}^k \rightarrow \{0,1\}^{k+1}$
  - $G(x) = f(x) \circ B(x)$
  - Where f: $\{0,1\}^k \rightarrow \{0,1\}^k$ is a one-way permutation, and B a hardcore predicate for f

    bijection
  - Claim: G is a PRG
    - For a random x, f(x) is also random (because permutation), and hence all of f(x) is next-bit unpredictable.
    - B is a hardcore predicate, so B(x) remains unpredictable after seeing f(x)

Summary

- OWF: a very simple cryptographic primitive with several candidates
- Every OWF/OWP has a hardcore predicate associated with it (Goldreich-Levin)
- PRG from a OWP and a hardcore predicate for it
  - A PRG can be constructed from a OWF too, but more complicated. (And, some candidate OWFs are anyway permutations.)
- Last time: PRF from PRG
- PRG can be used as a stream-cipher (for one-time CPA secure SKE), and a PRF can be used as a block-cipher (for full-fledged CPA secure SKE)