Public-Key Cryptography

Lecture 11

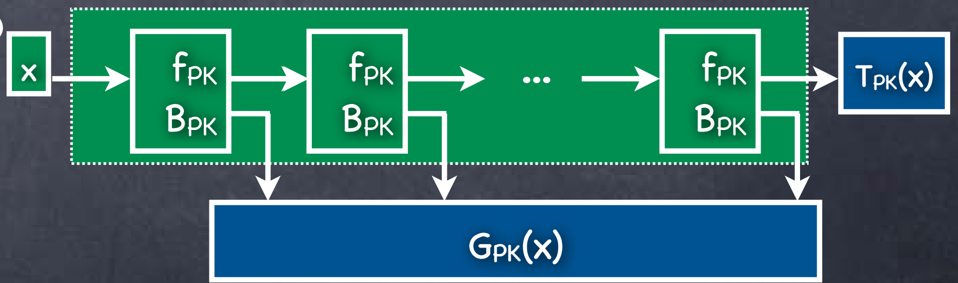
Some Trapdoor OWP Candidates

Chinese Remainder Theorem

RECALL

CPA-secure PKE for Trapdoor OWP

- CPA secure PKE from Trapdoor PRG
  - PRG family with a (PK,SK). PK specifies the family member.
  - Can encapsulate the seed for the PRG such that:
    - PRG output remains pseudorandom even given PK and encapsulated seed
    - Can recover PRG output from encapsulated seed and SK
  - El Gamal: encapsulated seed = g^x, PRG output = Y^x
- Trapdoor PRG from Trapdoor OWP



RECALL

Candidate Trapdoor OWPs

- Two candidates using composite moduli
  - RSA function: $f_{RSA}(x; N,e) = x^e \bmod N$ where N=PQ, P,Q k-bit primes, e s.t. $gcd(e,\phi(N)) = 1$ (and x uniform from {0...N-1})
    - Fact: $f_{RSA}(.; N,e)$ is a permutation
    - Fact: While picking (N,e), can also pick d s.t. $x^{ed} = x$
  - Rabin OWF: $f_{Rabin}(x; N) = x^2 \bmod N$, where N = PQ, and P, Q are k-bit primes (and x uniform from {0...N-1})
    - Fact: $f_{Rabin}(.; N)$ is a permutation among quadratic residues, when P, Q are $\equiv 3 \pmod 4$
    - Fact: Can invert $f_{Rabin}(.; N)$ given factorization of N

$\mathbb{Z}_N^*$

- Group operation: "multiplication modulo N"
  - Has identity, is associative
- Group elements: all numbers (mod N) which have a multiplicative inverse modulo N
  - e.g.: $\mathbb{Z}_6^*$ has elements {1,5}, $\mathbb{Z}_7^*$ has {1,2,3,4,5,6}
- a has a multiplicative inverse modulo N
  - $\Leftrightarrow$ $\exists$ integers b, c s.t. ab = 1+cN
  - $\Leftrightarrow$ gcd(a,N)=1 [Why?]
    - ($\Rightarrow$) gcd(a,N) | (ab-cN)
    - ($\Leftarrow$) from Euclid's algorithm: $\exists$ b, d s.t. gcd(a,N) = ab+dN
  - $|\mathbb{Z}_N^*|$ = #integers in [1,N-1] co-prime with N = $\phi(N)$

Extended Euclidean algorithm to find (b,d) given (a,N). Used to efficiently invert elements in $\mathbb{Z}_N^*$

$\mathbb{Z}_P^*$, P prime

- Recall $\mathbb{Z}_P^*$
- $|\mathbb{Z}_P^*|$ =: $\phi(P) = P-1$ (all of them co-prime with P)
- Cyclic: Isomorphic to $\mathbb{Z}_{P-1}$
- Discrete Log assumed to be hard
- Quadratic Residues form a subgroup $\mathbb{QR}_P^*$
  - $\mathbb{QR}_P^*$ is a candidate group for DDH assumption

$\mathbb{Z}_N^*$, N=PQ, two primes

Also works with P, Q co-primes

- e.g. $\mathbb{Z}_{15}^* = \{1,2,4,7,8,11,13,14\}$
  - $\phi(15) = 8$
- Group operation and inverse efficiently computable
- Cyclic?
  - No! In $\mathbb{Z}_{15}^*$, $2^4 = 4^2 = 7^4 = 8^4 = 11^2 = 13^4 = 14^2 = 1$ (i.e., each generates at most 4 elements, out of 8)
- "Product of two cycles": $\mathbb{Z}_3^*$ and $\mathbb{Z}_5^*$
  - Chinese Remainder Theorem

Chinese Remainder Theorem

- Consider mapping elements in $\mathbb{Z}_{15}$ (all 15 of them) to $\mathbb{Z}_3$ and $\mathbb{Z}_5$
  - $a \mapsto (a \bmod 3, a \bmod 5)$
- CRT says that the pair (a mod 3, a mod 5) uniquely determines a (mod 15)!
  - All 15 possible pairs occur, once each
- In general for N=PQ (P, Q relatively prime), $a \mapsto (a \bmod P, a \bmod Q)$ maps the N elements to the N distinct pairs
  - In fact extends to product of more than two (relatively prime) numbers

$\mathbb{Z}_{15}$	$\mathbb{Z}_3$	$\mathbb{Z}_5$
0	0	0
1	1	1
2	2	2
3	0	3
4	1	4
5	2	0
6	0	1
7	1	2
8	2	3
9	0	4
10	1	0
11	2	1
12	0	2
13	1	3
14	2	4

Chinese Remainder Theorem and $\mathbb{Z}_N$

- CRT representation of $\mathbb{Z}_N$: every element of $\mathbb{Z}_N$ can be written as a unique element of $\mathbb{Z}_P \times \mathbb{Z}_Q$
  - Addition can be done coordinate-wise
  - $(a,b) +_{(\text{mod } N)} (a',b') = (a +_{(\text{mod } P)} a', b +_{(\text{mod } Q)} b')$
- CRT: $\mathbb{Z}_N \cong \mathbb{Z}_P \times \mathbb{Z}_Q$ (group isomorphism)
- Can efficiently compute the isomorphism (in both directions) if P, Q known [Exercise]

$\mathbb{Z}_{15}$	$\mathbb{Z}_3$	$\mathbb{Z}_5$
0	0	0
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Chinese Remainder Theorem and $\mathbb{Z}_N^*$

- Elements in $\mathbb{Z}_N^*$
  - Consider the same mapping into $\mathbb{Z}_P \times \mathbb{Z}_Q$
  - Multiplication (and identity, and inverse) also coordinate-wise
  - No multiplicative inverse iff (0,b) or (a,0)
  - Else in $\mathbb{Z}_N^*$: i.e., (a,b) s.t. $a \in \mathbb{Z}_P^*$, $b \in \mathbb{Z}_Q^*$
    - $\mathbb{Z}_N^* \cong \mathbb{Z}_P^* \times \mathbb{Z}_Q^*$
  - $\phi(N) = |\mathbb{Z}_N^*| = (P-1)(Q-1)$ ($P \neq Q$, primes)

$\mathbb{Z}_{15}$	$\mathbb{Z}_3$	$\mathbb{Z}_5$
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RSA Function

- $f_{RSA[N,e]}(x) = x^e \bmod N$
  - Where N=PQ, and $\gcd(e,\phi(N)) = 1$ (i.e., $e \in \mathbb{Z}_{\phi(N)}^*$)
  - $f_{RSA[N,e]}: \mathbb{Z}_N \to \mathbb{Z}_N$
    - Alternately, $f_{RSA[N,e]}: \mathbb{Z}_N^* \to \mathbb{Z}_N^*$
- $f_{RSA[N,e]}$ is a permutation over $\mathbb{Z}_N$ with a trapdoor (namely (N,d))
  - In fact, there exists d s.t. $f_{RSA[N,d]}$ is the inverse of $f_{RSA[N,e]}$
    - d s.t. $ed \equiv 1 \pmod{\phi(N)} \Rightarrow x^{ed} \equiv x \pmod{N}$
    - Why? In $\mathbb{Z}_N^*$ because order of $\mathbb{Z}_N^*$ is $\phi(N)$
    - In $\mathbb{Z}_N$ too, by CRT: $\mathbb{Z}_N \cong \mathbb{Z}_P \times \mathbb{Z}_Q$ and $\phi(N) = \phi(P)\phi(Q)$
      - Exponentiation works coordinate-wise
      - $ed \equiv 1 \pmod{\phi(N)} \Rightarrow ed \equiv 1 \pmod{\phi(P)}$ and $ed \equiv 1 \pmod{\phi(Q)}$

RSA Function

- $f_{RSA[N,e]}(x) = x^e \bmod N$
  - Where N=PQ, and $\gcd(e,\phi(N)) = 1$ (i.e., $e \in \mathbb{Z}_{\phi(N)}^*$)
  - $f_{RSA[N,e]}: \mathbb{Z}_N \rightarrow \mathbb{Z}_N$
    - Alternately, $f_{RSA[N,e]}: \mathbb{Z}_N^* \rightarrow \mathbb{Z}_N^*$
- $f_{RSA[N,e]}$ is a permutation over $\mathbb{Z}_N$ with a trapdoor (namely (N,d))
- RSA Assumption: $f_{RSA[N,e]}$ is a OWF collection, when P, Q random k-bit primes and e < N random number s.t. $\gcd(e,\phi(N))=1$ (with inputs uniformly from $\mathbb{Z}_N$ or $\mathbb{Z}_N^*$)
  - Alternate version: e=3, P, Q restricted so that $\gcd(3,\phi(N))=1$
- RSA Assumption will be false if one can factorize N
  - Then knows $\phi(N) = (P-1)(Q-1)$ and can find d s.t. $ed \equiv 1 \pmod{\phi(N)}$
  - Converse not known to hold
- Trapdoor OWP Candidate

Rabin Function

- $f_{Rabin[N]}(x) = x^2 \bmod N$ where N=PQ, P,Q primes $\equiv 3 \bmod 4$
  - Is a candidate OWF collection (indexed by N)
    - Equivalent to the assumption that f_{mult} is a OWF (for the appropriate distribution)
      - If can factor N, will see how to find square-roots
        - So (P,Q) a trapdoor to "invert"
      - Fact: If can take square-root mod N, can factor N
  - Coming up: Is a permutation over $\mathbb{QR}_N^*$, with trapdoor (P,Q)

Square-roots in $\mathbb{Z}_P^*$

- What are the square-roots of x^2 mod a prime?
- $\sqrt{1} = \pm 1$
  - $x^2=1 \pmod P \Leftrightarrow (x+1)(x-1) = 0 \pmod P$

    $\Leftrightarrow (x+1)=0$ or $(x-1)=0 \pmod P$

    $\Leftrightarrow x=1 \pmod P$ or $x=-1 \pmod P$

    P is prime
  - Where $-1 = g^{(P-1)/2}$
- More generally $\sqrt{(x^2)} = \pm x$ (because $x^2=y^2 \pmod P \Leftrightarrow x = \pm y$)

Square-roots in $\mathbb{Z}_p^*$

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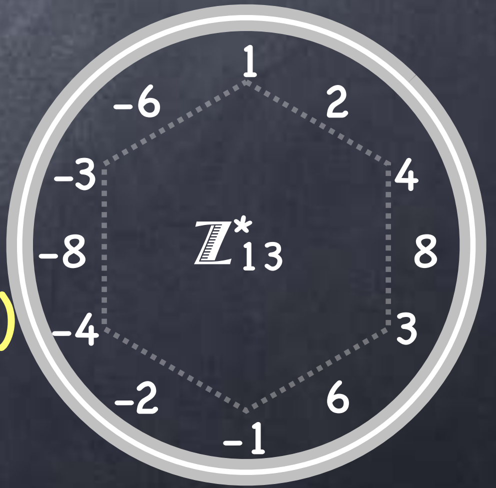
    $\Leftrightarrow x=1 \pmod{P}$ or $x=-1 \pmod{P}$
  - Where $-1 = g^{(P-1)/2}$
- More generally $\sqrt{(x^2)} = \pm x$ (because $x^2=y^2 \pmod{P} \Leftrightarrow x = \pm y$)
  - $-x = x \cdot g^{(P-1)/2}$ appears "diametrically opposite" x

Square-roots in $\mathbb{QR}_p^*$

- In $\mathbb{Z}_p^*$ $\sqrt{(x^2)} = \pm x$
- How many square-roots are in $\mathbb{QR}_p^*$?
  - Depends on P!
  - e.g. $\mathbb{QR}_{13}^* = \{\pm 1, \pm 3, \pm 4\}$
    - 1,3,-4 have 2 square-roots each. But -1,-3,4 have none within $\mathbb{QR}_{13}^*$
    - Since $-1 \in \mathbb{QR}_{13}^*$, $x \in \mathbb{QR}_{13}^* \Rightarrow -x \in \mathbb{QR}_{13}^*$
    - $-1 \in \mathbb{QR}_p^*$ iff $(P-1)/2$ even
- If $(P-1)/2$ odd, exactly one of $\pm x$ in $\mathbb{QR}_p^*$ (for all x)
  - Then, squaring is a permutation in $\mathbb{QR}_p^*$



Square-roots in $\mathbb{QR}_P^*$

- In $\mathbb{Z}_P^*$ $\sqrt{(x^2)} = \pm x$ (i.e., x and $-1 \cdot x$)
- If $(P-1)/2$ odd, squaring is a permutation in $\mathbb{QR}_P^*$
  - $(P-1)/2$ odd $\Leftrightarrow$ $P \equiv 3 \pmod 4$

- But easy to compute both ways!
  - In fact $\sqrt{z} = z^{(P+1)/4} \in \mathbb{QR}_P^*$ (because $(P+1)/2$ even)
- Rabin function defined in $\mathbb{QR}_N^*$ and relies on keeping the factorization of $N=PQ$ hidden

$\mathbb{QR}_N^*$

- What do elements in $\mathbb{QR}_N^*$ look like, for N=PQ?
  - By CRT, can write $a \in \mathbb{Z}_N^*$ as $(x,y) \in \mathbb{Z}_P^* \times \mathbb{Z}_Q^*$
  - CRT representation of a^2 is $(x^2,y^2) \in \mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
  - $\mathbb{QR}_N^* \cong \mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
  - If both $P,Q \equiv 3 \pmod 4$, then squaring is a permutation in $\mathbb{QR}_N^*$
    - $\sqrt{(x^2,y^2)} = (\pm x, \pm y)$ in $\mathbb{Z}_P^* \times \mathbb{Z}_Q^*$ but exactly one in $\mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
    - Can efficiently do this, if can compute (and invert) the isomorphism from $\mathbb{QR}_N^*$ to $\mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
      - (P,Q) is a trapdoor
    - Without trapdoor, OWF candidate
      - Follows from assuming squaring is a OWF over the domain $\mathbb{Z}_N^*$, because $\mathbb{QR}_N^*$ forms 1/4th of $\mathbb{Z}_N^*$

Rabin Function

- $f_{Rabin[N]}(x) = x^2 \bmod N$
  - Candidate OWF collection, with N=PQ (P,Q random k-bit primes)
  - If P, Q $\equiv 3 \pmod 4$, then, restricted to $\mathbb{QR}_N^*$:
    - A permutation
    - Has a trapdoor for inverting (namely (P,Q))
- Candidate Trapdoor OWP

Can sample efficiently by sampling $x \leftarrow \mathbb{Z}_N^*$, and outputting x^2

Summary

- A DLA candidate: $\mathbb{Z}_P^*$
- A DDH candidate: $\mathbb{QR}_P^*$ where P is a safe prime
- Chinese Remainder Theorem
  - $\mathbb{Z}_N \cong \mathbb{Z}_P \times \mathbb{Z}_Q$
  - $\mathbb{Z}_N^* \cong \mathbb{Z}_P^* \times \mathbb{Z}_Q^*$
  - $\mathbb{QR}_N^* \cong \mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
- Trapdoor OWP candidates:
  - $f_{RSA[N,e]} = x^e \bmod N$ where N=PQ and $\gcd(e,\phi(N))=1$
    - Trapdoor: $(P,Q) \rightarrow \phi(N) \rightarrow d=e^{-1}$ in $\mathbb{Z}^*_{\phi(N)}$
  - $f_{Rabin[N]} = x^2 \bmod N$ where N=PQ, where $P,Q \equiv 3 \pmod 4$ (restricted to $\mathbb{QR}_N^*$)
    - Trapdoor: (P,Q)
- Trapdoor OWP can be used to construct Trapdoor PRG
  - Trapdoor PRG can give IND-CPA secure PKE