Hash Functions (ctd.)

Lecture 15

RECALL

Hash Functions

- Main syntactic feature: Variable input length to fixed length output
- Primary requirement: collision-resistance
  - If for all PPT A, Pr[x≠y and h(x)=h(y)] is negligible in the following experiment:
    - $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$: Combinatorial Hash Functions
    - $A \rightarrow x$; $h \leftarrow \mathcal{H}$; $A(h) \rightarrow y$: Universal One-Way Hash Functions
    - $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$: Collision-Resistant Hash Functions
      - $h \leftarrow \mathcal{H}$; $A^h \rightarrow (x,y)$: Weak Collision-Resistant Hash Functions
    - $h \leftarrow \mathcal{H}$; $x \leftarrow X$; $A(h,h(x)) \rightarrow y$: One-Way Hash Functions (x=y OK)
    - $h \leftarrow \mathcal{H}$; $x \leftarrow X$; $A(h,x) \rightarrow y$: SPR Hash Functions

Typically used

- Also often required: "unpredictability"
- Already saw: a 2-UHF (chop(ax+b) over a field)
- Today: UOWHF and CRHF constructions. Domain Extension.

Generalizes to vector spaces [Exercise]

UOWHF

- Universal One-Way HF: $A \rightarrow x$; $h \leftarrow \mathcal{H}$; $A(h) \rightarrow y$. $h(x)=h(y)$ w.n.p
- Since the hash function is compressing, then there will be collisions. So a computationally unbounded adversary can win this game!
- Need to rely on computational hardness
- UOWHF can be constructed from OWF
- Much easier to see OWP $\Rightarrow$ UOWHF

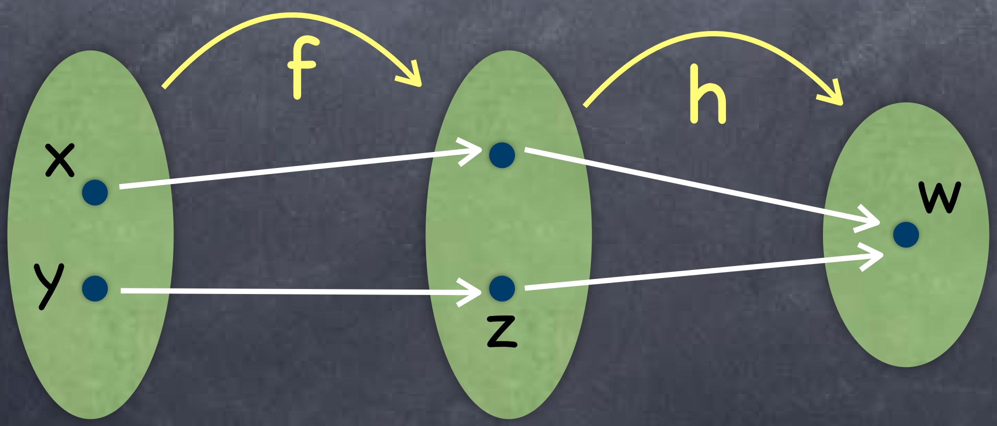
UOWHF from OWP

Or, if not unique, can uniformly sample a solution for h

- $F_h(x) = h(f(x))$, where f is a OWP and h from a 2-UHF family
  - s.t. h is a 2-to-1 map, and
  - for all z, z′, w, can efficiently solve for h s.t. $h(z) = h(z') = w$
- Is a UOWHF: can choose h to force UOWHF adversary to invert f

BreakOWP(z) { Get $x \leftarrow A$; Sample random w; Solve h s.t. $h(z) = h(f(x)) = w$;
Give h to A; Get $y \leftarrow A$ and output y; }

  - Only collision ($y \neq x$ s.t. $F_h(x) = F_h(y)$) is $y = f^{-1}(z)$



UOWHF from OWP

Or, if not unique, can uniformly sample a solution for h

- $F_h(x) = h(f(x))$, where f is a OWP and h from a 2-UHF family
  - s.t. h is a 2-to-1 map, and
  - for all z, z′, w, can efficiently solve for h s.t. $h(z) = h(z') = w$
- Is a UOWHF: can choose h to force UOWHF adversary to invert f

  BreakOWP(z) { Get x ← A; Sample random w; Solve h s.t. h(z) = h(f(x)) = w; Give h to A; Get y ← A and output y; }

  - Only collision ($y \neq x$ s.t. $F_h(x) = F_h(y)$) is $y=f^{-1}(z)$
  - BreakOWP is efficient as h can be efficiently solved ✓
  - BreakOWP has same advantage as A has against UOWHF? Yes, if h is uniform (independent of x) [Why?]
    - h uniform because z, w picked uniformly ✓

CRHF

- Collision-Resistant HF: $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$. $h(x)=h(y)$ w.n.p
- Not known to be possible from OWF/OWP alone
  - "Impossibility" (blackbox-separation) known
- Possible from "claw-free pair of permutations"
  - In turn from hardness of discrete-log, factoring, and from lattice-based assumptions
- Also from "homomorphic one-way permutations", and from homomorphic encryptions
  - All candidates use mathematical operations that are considered computationally expensive

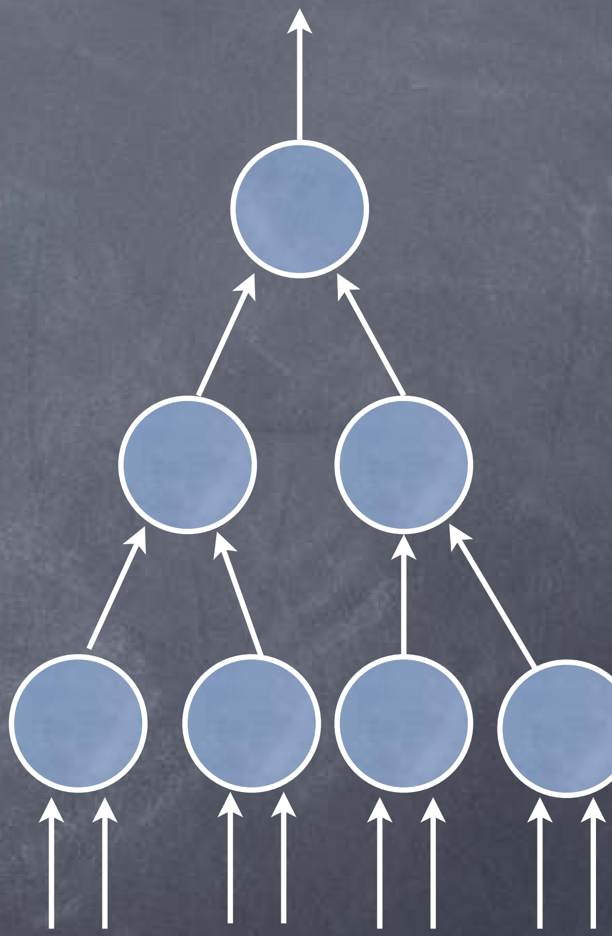
CRHF

- CRHF from discrete log assumption:
  - Suppose $\mathbb{G}$ a group of prime order q, where DL is considered hard (e.g. $\mathbb{QR}_p^*$ for p=2q+1 a safe prime)
  - $h_{g1,g2}(x_1,x_2) = g_1^{x1}g_2^{x2}$ (in $\mathbb{G}$) where $g_1, g_2 \neq 1$ (hence generators)
  - A collision: $(x_1,x_2) \neq (y_1,y_2)$ s.t. $h_{g1,g2}(x_1,x_2) = h_{g1,g2}(y_1,y_2)$
    - Collision $\Rightarrow$ $x_1 \neq y_1$ and $x_2 \neq y_2$ [Why?]
    - Then $g_2 = g_1^{(x1-y1)/(x2-y2)}$ (exponents in $\mathbb{Z}_q^*$)
      - i.e., w.r.t. a random base g_1, can compute DL of a random element g_2. Breaks DL!
  - Hash halves the size of the input

Domain Extension

- Full-domain hash: hash arbitrarily long strings to a single hash value
  - So far, UOWHF/CRHF which have a fixed domain
- First, simpler goal: extend to a larger, fixed domain
  - Assume we are given a hash function from two blocks to one block (a block being, say, k bits)
    - What if we can compress by only one bit (e.g., our UOWHF construction)?
      - Can just apply repeatedly to compress by t bits

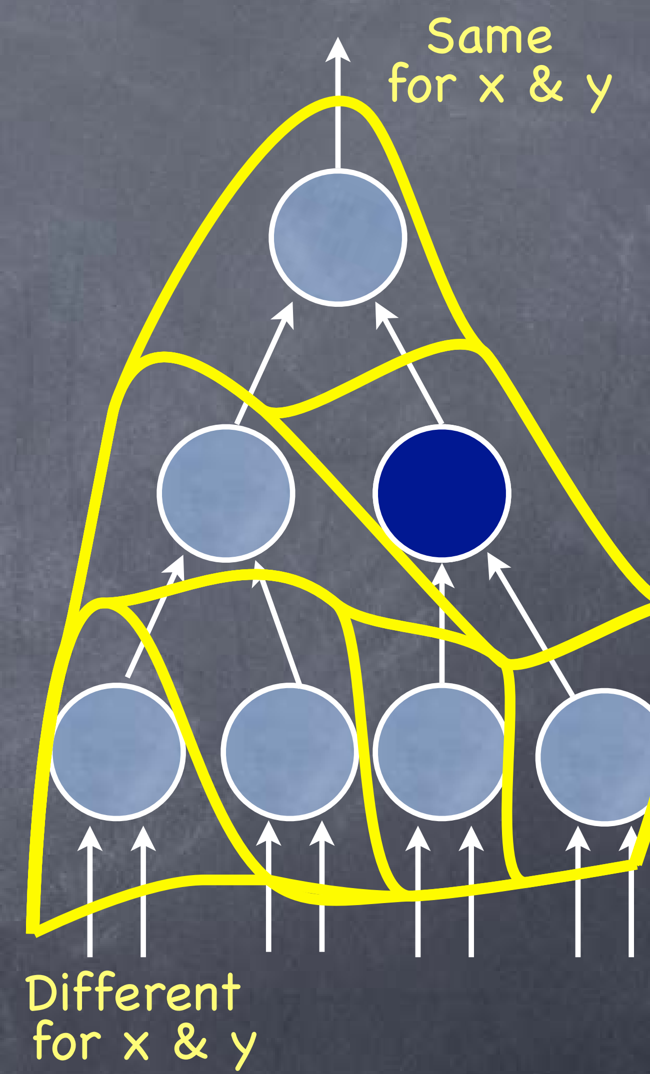
Domain Extension



- Can compose hash functions more efficiently, using a "**Merkle tree**"
  - Uses a basic hash from $\{0,1\}^{2k}$ to $\{0,1\}^k$
  - Example: A hash function from $\{0,1\}^{8k}$ to $\{0,1\}^k$ using a tree of depth 3
  - Any tree can be used, with consistent I/O sizes
- Independent hashes or same hash?
  - Depends!

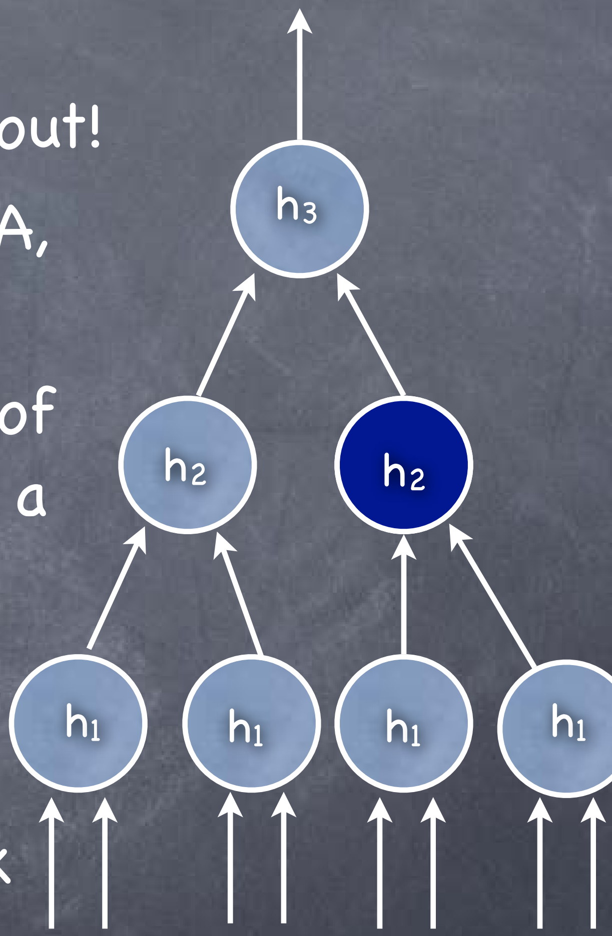
Domain Extension for CRHF

- For CRHF, same basic hash used through out the Merkle tree. Hash description same as for a single basic hash
- If a collision ($(x_1 \ldots x_n)$, $(y_1 \ldots y_n)$) over all, then some collision (x',y') for basic hash
    - Consider moving a "frontline" from bottom to top. Look for equality on this front.
        - Collision at some step (different values on i^{th} front, same on $i+1^{st}$); gives a collision for basic hash
- $A^*(h)$: run $A(h)$ to get $(x_1 \ldots x_n)$, $(y_1 \ldots y_n)$. Move frontline to find (x',y')



Domain Extension for UOWHF

- For UOWHF, can't use same basic hash throughout!
- A^* has to output an x' on getting $(x_1 \ldots x_n)$ from A, before getting h
  - Can guess a random node (i.e., random pair of frontlines) where collision occurs, but if not a leaf, can't compute x' until h is fixed!
- Solution: a different h for each level of the tree (i.e., no ancestor/successor has same h)
  - To compute x': Get $(x_1 \ldots x_n)$ from A. Then pick a random node (say at level i), pick h_j for levels below i, and compute input to the node; let this be x'.
  - On getting h, plug it in as h_i, pick h_j for remaining levels; give h's to A and get $(y_1 \ldots y_n)$; compute y' and output it.



UOWHF vs. CRHF

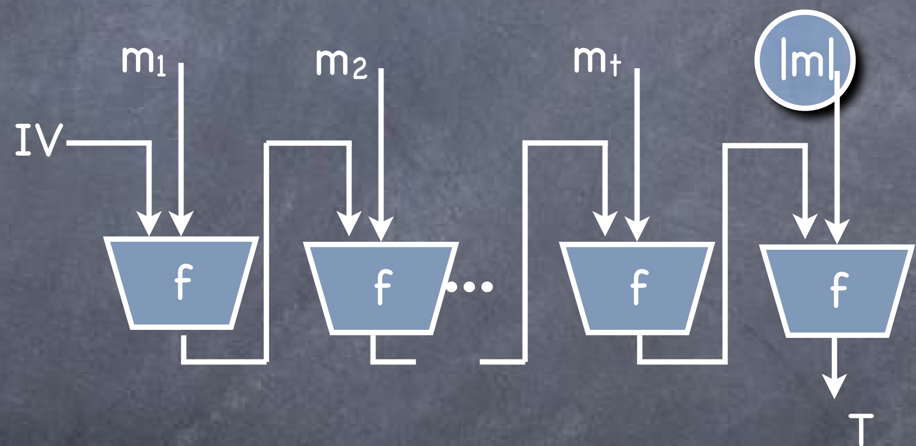
- UOWHF has a weaker guarantee than CRHF
- UOWHF can be built based on OWF (we saw based on OWP), where as CRHF "needs stronger assumptions"
  - But "usual" OWF candidates suffice for CRHF too (we saw construction based on discrete-log)
- Domain extension of CRHF is simpler, with no blow-up in the description size. For UOWHF description increases logarithmically in the input size
- UOWHF theoretically important (based on simpler assumptions, good if paranoid), but CRHF can substitute for it
- Current practice: much less paranoid; faith on efficient, ad hoc (and unkeyed) constructions

Domain Extension

- Full-domain hash: hash arbitrarily long strings to a single hash value
  - Merkle-Tree construction extends the domain to any fixed input length
- Hash the message length (number of blocks) along with the original hash
  - Collision in the new hash function gives either collision at the top level, or if not, collision in the original Merkle tree and <u>for the same message length</u>

Hash Functions in Practice

- A single function, not a family (e.g. SHA-3, SHA-256, MD4, MD5)
- Often from a fixed input-length compression function
- Merkle-Damgård iterated hash function, MD^f:



Collision resistance even with variable input-length.

Note: Unlike MACs, here "length-extension" is OK, as long as it results in a different hash value

If f is not keyed, but "concretely" collision resistant, so is MD^f

- If f collision resistant then so is MD^f (for any IV)
  - If f modelled as a Random Oracle, MD^f is a "public-use RO." If f modelled as an "Ideal Cipher," MD^f is "pre-image aware."