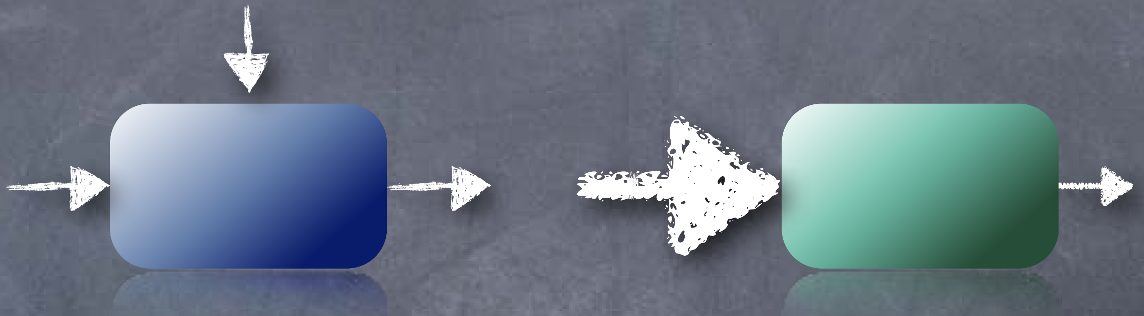
Hash Functions

Lecture 10

Flavours of collision resistance

A Tale of Two Boxes

- The bulk of today's applied cryptography works with two magic boxes
  - Block Ciphers
  - Hash Functions



- Block Ciphers: Best modeled as (strong) Pseudorandom Permutations, with inversion trapdoors
  - Often more than needed (e.g. SKE needs only PRF)
- Hash Functions:
  - Some times modelled as Random Oracles!
    - Use at your own risk! No guarantees in the standard model.
  - Today: understanding security requirements on hash functions

Hash Functions

- "Randomised" mapping of inputs to shorter hash-values
- Hash functions are useful in various places
  - In data-structures: for efficiency
    - Intuition: hashing removes worst-case effects
  - In cryptography: for "integrity"
- Primary use: Domain extension (compress long inputs, and feed them into boxes that can take only short inputs)
  - Typical security requirement: "collision resistance"
    - Different flavours: some imply one-wayness
  - Also sometimes: some kind of unpredictability

Hash Function Family

- Hash function $h:\{0,1\}^{n(k)} \rightarrow \{0,1\}^{t(k)}$
  - Compresses
- A family
  - Alternately, takes two inputs, the index of the member of the family, and the real input
- Efficient sampling and evaluation
- Idea: when the hash function is randomly chosen, "behaves randomly"
  - Main goal: to "avoid collisions". Will see several variants of the problem

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$	...	$h_N(x)$
000	0	0	0	1	...	1
001	0	0	1	1		1
010	0	1	0	1		1
011	0	1	1	0		1
100	1	0	0	1		1
101	1	0	1	0		1
110	1	1	0	1		1
111	1	1	1	0		1

Hash Functions in Crypto Practice

- A single fixed function
  - e.g. SHA-3, SHA-256, SHA-1, MD5, MD4
  - Not a family ("unkeyed")
  - (And no security parameter knob)
- Not collision-resistant under any of the following definitions
- Alternately, could be considered as having already been randomly chosen from a family (and security parameter fixed too)
  - Usually involves hand-picked values (e.g. "I.V." or "round constants") built into the standard

Degrees of Collision-Resistance

- If for all PPT A, Pr[$x \neq y$ and $h(x)=h(y)$] is negligible in the following experiment:
  - $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$: Combinatorial Hash Functions (even non-PPT A)
  - $A \rightarrow x$; $h \leftarrow \mathcal{H}$; $A(h) \rightarrow y$: Universal One-Way Hash Functions
  - $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$: Collision-Resistant Hash Functions
- CRHF the strongest. UOWHF of theoretical interest (powerful enough for digital signatures, and can be based on OWF alone).
- Useful variants: A gets only oracle access to $h(\cdot)$ (weaker). Or, A gets any coins used for sampling h (stronger).

Degrees of Collision-Resistance

- Variants of CRHF where x is random
  - $h \leftarrow \mathcal{H}$; $x \leftarrow X$; $A(h,h(x)) \rightarrow y$ ($y=x$ allowed)
    - Pre-image collision resistance if $h(x)=h(y)$ w.n.p
    - i.e., $f(h,x) := (h,h(x))$ is a OWF (and h compresses)
  - $h \leftarrow \mathcal{H}$; $x \leftarrow X$; $A(h,x) \rightarrow y$ ($y \neq x$)
    - Second Pre-image collision resistance if $h(x)=h(y)$ w.n.p
  - Incomparable (neither implies the other) [Exercise]
- CRHF implies second pre-image collision resistance and, if compressing, then pre-image collision resistance [Exercise]

A.k.a One-Way Hash Function

Hash Length

- If range of the hash function is too small, not collision-resistant
  - If range poly(k)-size (i.e. hash is logarithmically long), then non-negligible probability that two random x, y provide collision
- In practice interested in minimising the hash length (for efficiency)
  - Generic attack on a CRHF: birthday attack
    - Look for a collision in a set of random inputs (needs only oracle access to the hash function)
      - Expected size of the set before collision: $O(\sqrt{|range|})$
  - Birthday attack effectively halves the security (hash length) of a CRHF compared to a generic attack on UOWHF

Universal Hashing

- Combinatorial HF: $A \rightarrow (x,y)$; $h \leftarrow \mathcal{H}$. $h(x)=h(y)$ w.n.p
- Even better: 2-Universal Hash Functions
  - "Uniform" and "Pairwise-independent"
  - $\forall x,z\ \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=z\ ] = 1/|Z|$ (where $h:X \rightarrow Z$)
  - $\forall x \neq y,w,z\ \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=w,\ h(y)=z\ ] = \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=w\ ] \cdot \Pr_{h \leftarrow \mathcal{H}}[\ h(y)=z\ ]$
    - $\Rightarrow \forall x \neq y\ \Pr_{h \leftarrow \mathcal{H}}[\ h(x)=h(y)\ ] = 1/|Z|$

Negligible collision-probability if super-polynomial-sized range

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$
0	0	0	1	1
1	0	1	0	1
2	1	0	0	1

- k-Universal:
  - $\forall x_1..x_k$ (distinct), $z_1..z_k$, $\Pr_{h \leftarrow \mathcal{H}}[\forall i\ h(x_i)=z_i\ ] = 1/|Z|^k$
- Inefficient example: $\mathcal{H}$ set of all functions from X to Z
  - But we will need all $h \in \mathcal{H}$ to be succinctly described and efficiently evaluable

Universal Hashing

- Combinatorial HF: A→(x,y); $h \leftarrow \mathcal{H}$. h(x)=h(y) w.n.p
- Even better: 2-Universal Hash Functions
  - "Uniform" and "Pairwise-independent"
  - $\forall x \neq y, w, z$ $\Pr_{h \leftarrow \mathcal{H}}$ [h(x)=w, h(y)=z] = $1/|Z|^2$
    - $\Rightarrow \forall x \neq y$ $\Pr_{h \leftarrow \mathcal{H}}$ [h(x)=h(y)] = $1/|Z|$

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$
0	0	0	1	1
1	0	1	0	1
2	1	0	0	1

Negligible collision-probability if super-polynomial-sized range

- e.g. $h_{a,b}(x) = ax+b$ (in a finite **field**, X=Z)
  - Uniform
    - $\Pr_{a,b}$ [ax+b = z] = $\Pr_{a,b}$ [b = z-ax] = $1/|Z|$
  - $\Pr_{a,b}$ [ax+b = w, ay+b = z] = ? In a field, exactly one (a,b) satisfying the two equations (for $x \neq y$)
    - $\Pr_{a,b}$ [ax+b = w, ay+b = z] = $1/|Z|^2$
  - But does not compress!

Universal Hashing

- Combinatorial HF: A→(x,y); h←$\mathcal{H}$. h(x)=h(y) w.n.p
- Even better: 2-Universal Hash Functions
  - "Uniform" and "Pairwise-independent"
  - $\forall x \neq y, w, z$ $\Pr_{h \leftarrow \mathcal{H}}$ [h(x)=w, h(y)=z] = $1/|Z|^2$
    - ⇒ $\forall x \neq y$ $\Pr_{h \leftarrow \mathcal{H}}$ [h(x)=h(y)] = $1/|Z|$

x	$h_1(x)$	$h_2(x)$	$h_3(x)$	$h_4(x)$
0	0	0	1	1
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2	1	0	0	1

Negligible collision-probability if super-polynomial-sized range

- e.g. Chop(h(x)) where
  - h from a (possibly non-compressing) 2-universal HF
  - Chop a t-to-1 map from Z to Z'
  - e.g. with $|Z|=2^k$, removing last bit gives a 2-to-1 mapping
    - $\Pr_h$ [Chop(h(x)) = w, Chop(h(y)) = z]
      = $\Pr_h$ [h(x) = w0 or w1, h(y) = z0 or z1] = $4/|Z|^2 = 1/|Z'|^2$

Cryptographic Hash Functions

- Combinatorial collision resistance depended on the hash function being randomly chosen after (independent of) adversary's pair (x,y)
- But if the hash function is known first, adversary can find collisions
- Often the hash function does have to be public
- Solution: OK if finding collisions is computationally infeasible
  - Cryptographic hash-functions
    - CRHF (and UOWHF)

CRHF: In Theory

- Collision-Resistant HF: $h \leftarrow \mathcal{H}$; $A(h) \rightarrow (x,y)$. $\forall$PPT A, h(x)=h(y) w.n.p
- Not known to be possible from OWF/OWP alone
  - "Impossibility" (blackbox-separation) known
- Possible from "claw-free pair of permutations"
  - In turn from hardness of discrete-log, factoring, and from lattice-based assumptions
- Also from "homomorphic one-way permutations", and from homomorphic encryptions
- These candidates use mathematical operations that are fairly expensive (comparable to public-key encryption)

CRHF: In Theory

- CRHF from discrete log assumption:
  - Suppose $\mathbb{G}$ a group of prime order q, where DL is considered hard (e.g. $\mathbb{QR}_p^*$ for p=2q+1 a safe prime — i.e., q prime)
  - $h_{g1,g2}(x_1,x_2) = g_1^{x1} g_2^{x2}$ (in $\mathbb{G}$) where g_1, $g_2 \neq 1$ (hence generators)
  - A collision: $(x_1,x_2) \neq (y_1,y_2)$ s.t. $h_{g1,g2}(x_1,x_2) = h_{g1,g2}(y_1,y_2)$
    - Collision $\Rightarrow$ $x_1 \neq y_1$ and $x_2 \neq y_2$ [Why?]
    - Then $g_2 = g_1^{(x1-y1)/(x2-y2)}$ (exponents in $\mathbb{Z}_q^*$)
      - i.e., w.r.t. a random base g_1, can compute DL of a random element g_2. Breaks DL!
  - Hash halves the size of the input

Today

- Combinatorial hash functions, UOWHF and CRHF
  - (And weaker variants of CRHF: pre-image collision resistance and second-pre-image collision resistance)
- Collision-resistant combinatorial HF from 2-Universal Hash Functions
- A candidate CRHF construction based on Discrete Log assumption
- Coming up
  - Domain extension: Merkle Tree, Merkle-Damgård iterated hash