Zero Knowledge Proofs

Lecture 13

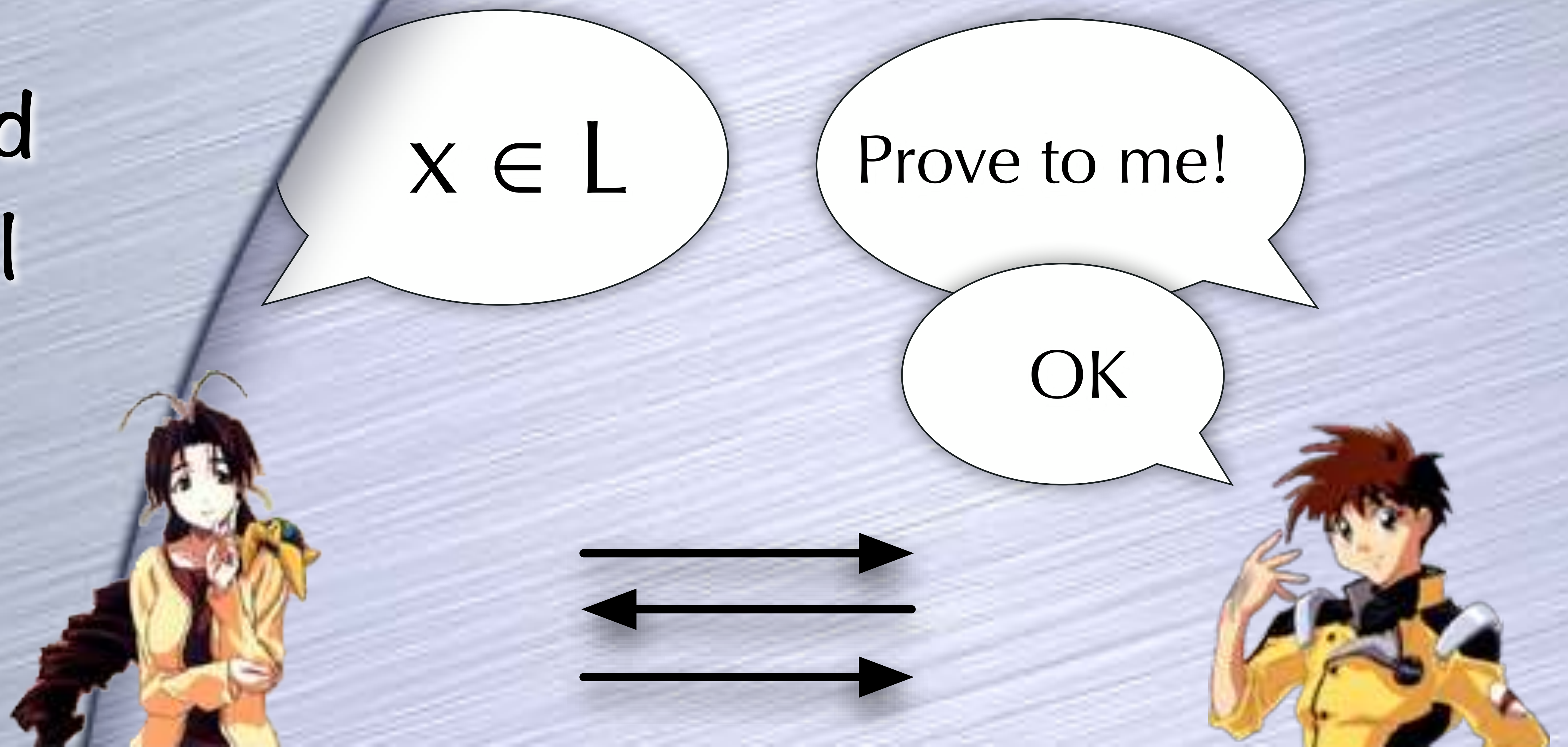
A Detour

Digital Signatures from Proof Systems

- Digital signatures can be seen as a proof of possession of a secret (signing) key, where the proof is tied with a message in a non-malleable fashion
  - Unforgeability: Seeing a proof tied to one message shouldn't leak the key, or enable one to give a proof of possessing it tied to another message
- It turns out that "proof systems" can indeed be turned into signature schemes
  - In the random oracle model, these form the basis of some of the most standard signature systems (DSA/ECDSA, EdDSA)
- Today
  - Interactive proof systems
    - Eventually, to be useful as a digital signature, we will need a non-interactive proof.
  - Zero-Knowledge proof systems
    - Helps in ensuring that the signatures don't leak the signing key

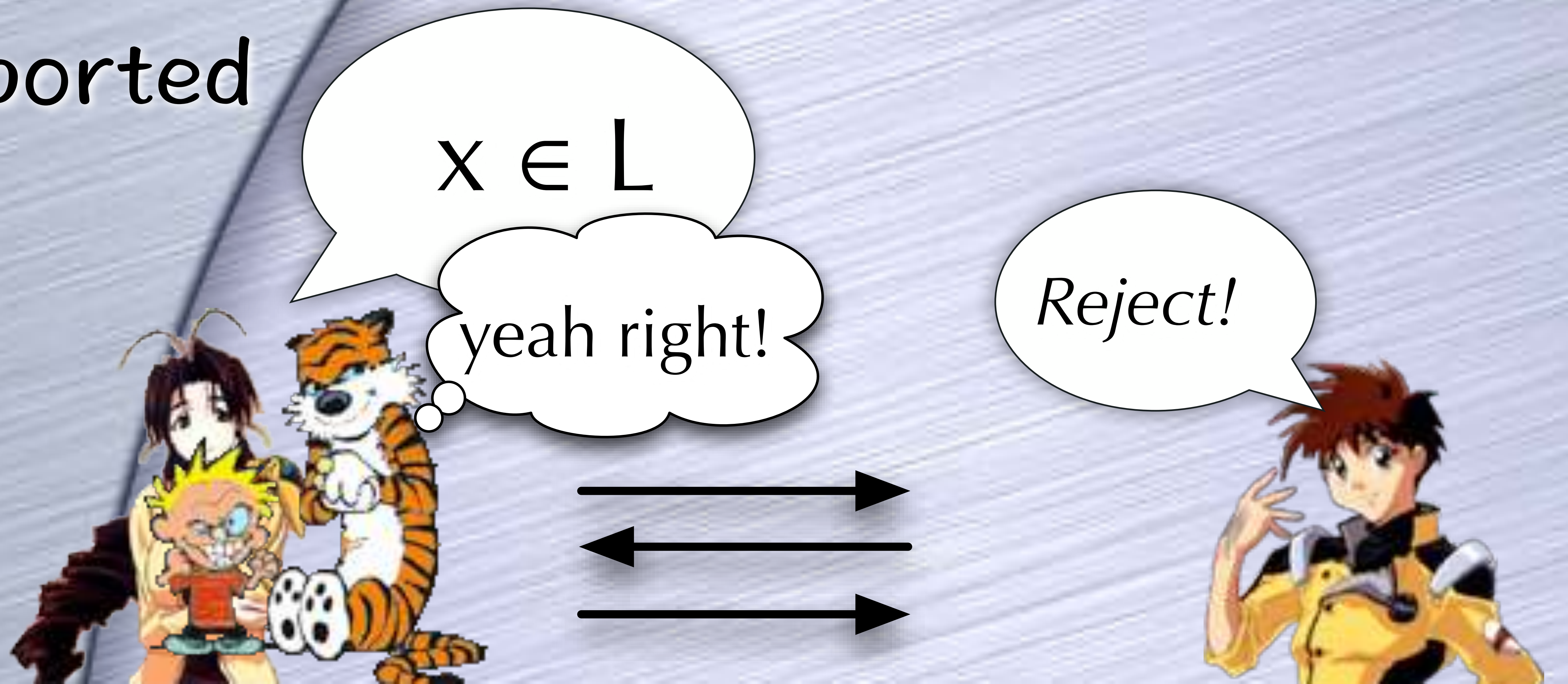
Interactive Proofs

- **Prover** wants to convince **verifier** that x has some property
  - i.e. x belongs to some set L ("language" L)
- Computationally bounded verifier, but all powerful prover (for now)



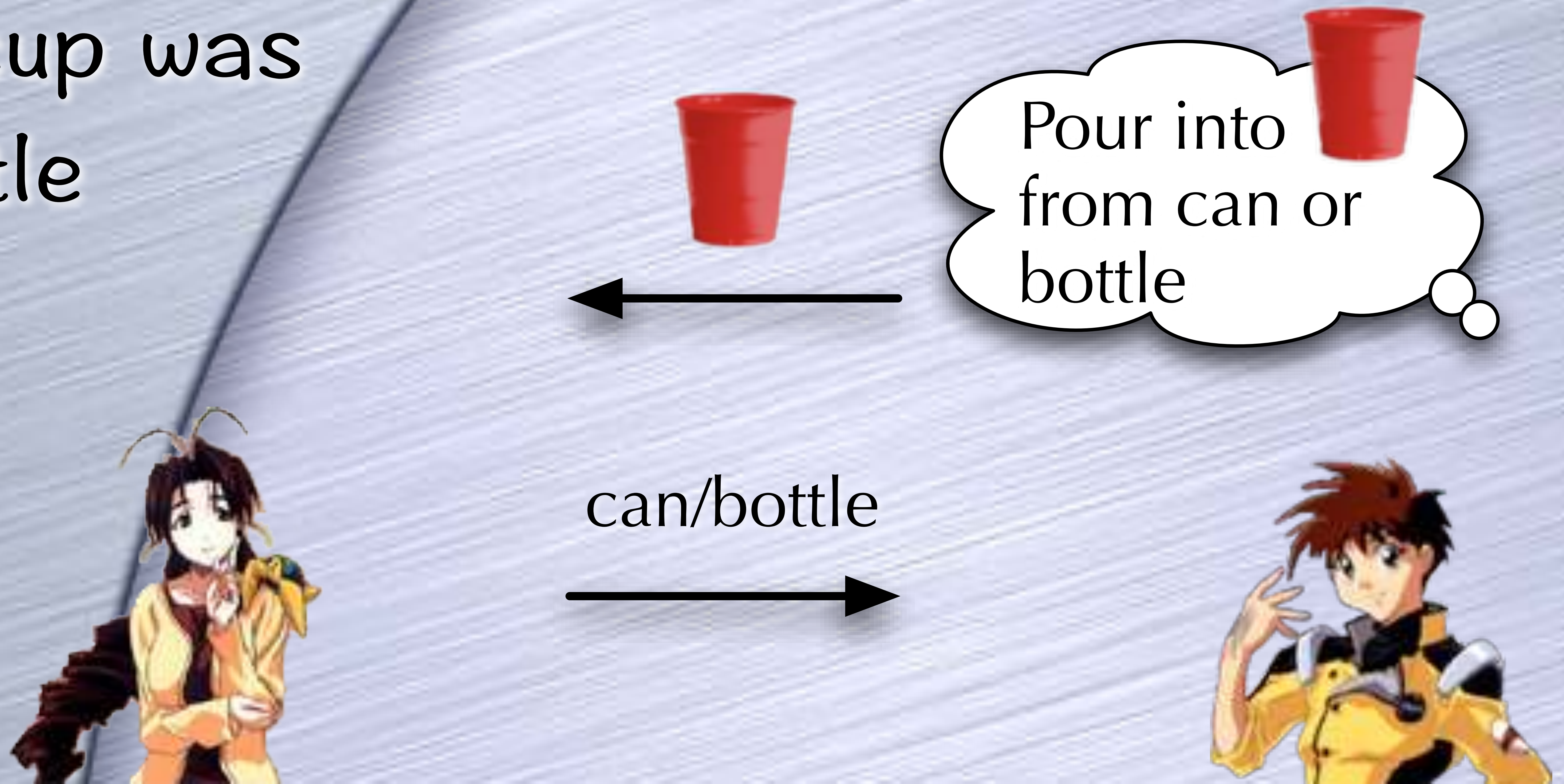
Interactive Proofs

- Completeness
  - If x in L, honest Prover will convince honest Verifier
- Soundness
  - If x not in L, honest Verifier won't accept any purported proof



An Example

- Coke in bottle or can
  - Prover claims: coke in bottle and coke in can are different
- IP protocol:
  - prover tells whether cup was filled from can or bottle
  - repeat till verifier is convinced



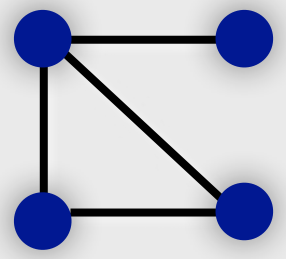
An Example

- Graph Non-Isomorphism
  - Prover claims: G_0 not isomorphic to G_1
- IP protocol:
  - prover tells whether G^* is an isomorphism of G_0 or G_1
  - repeat till verifier is convinced

Isomorphism: Same graph can be represented as a matrix in different ways:

e.g. $G_0 = \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$ and $G_1 = \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$

both are isomorphic to the graph represented by the drawing



Set G^* to be $\pi(G_0)$ or $\pi(G_1)$ (π random)

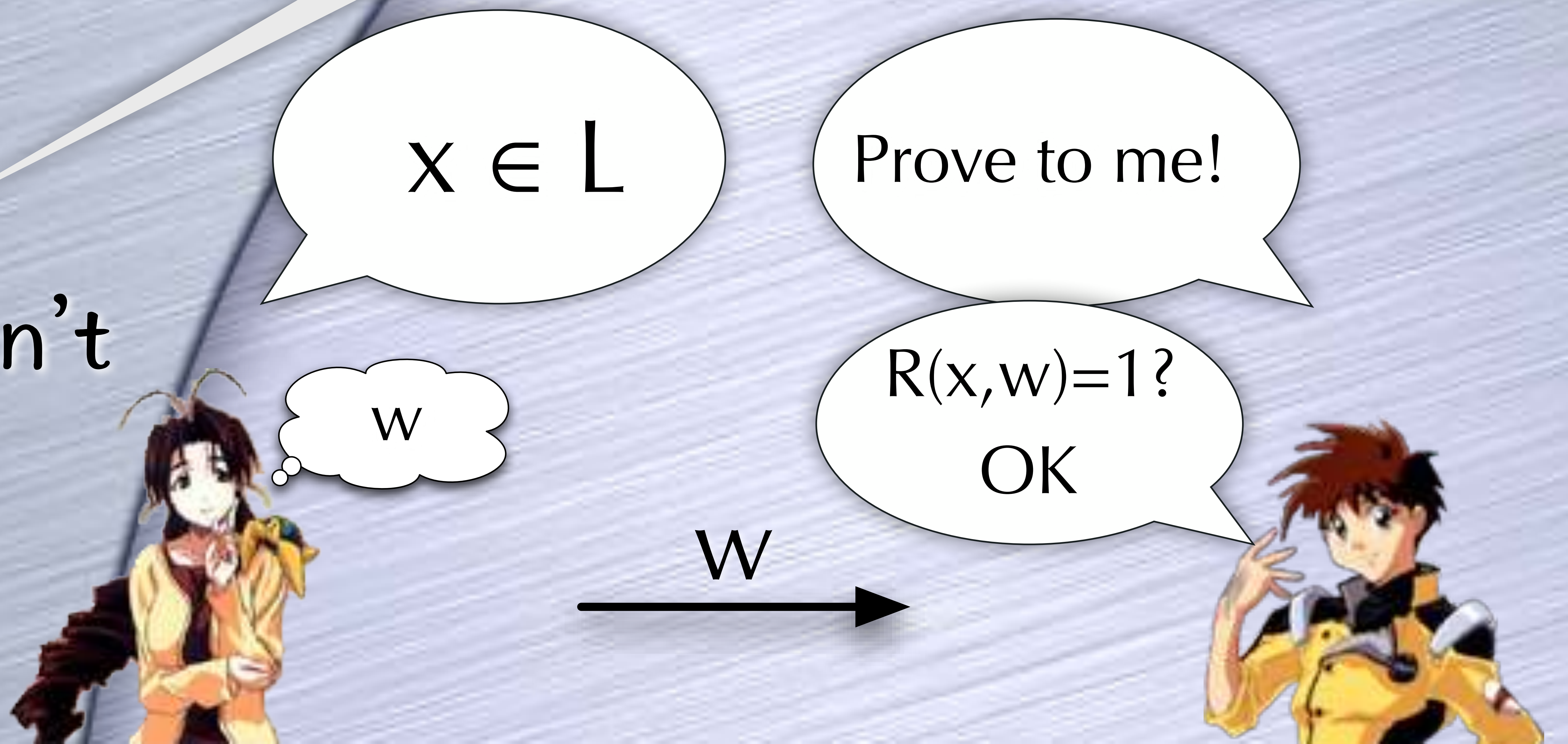
G^* ←

G_0/G_1 →

Proofs for NP languages

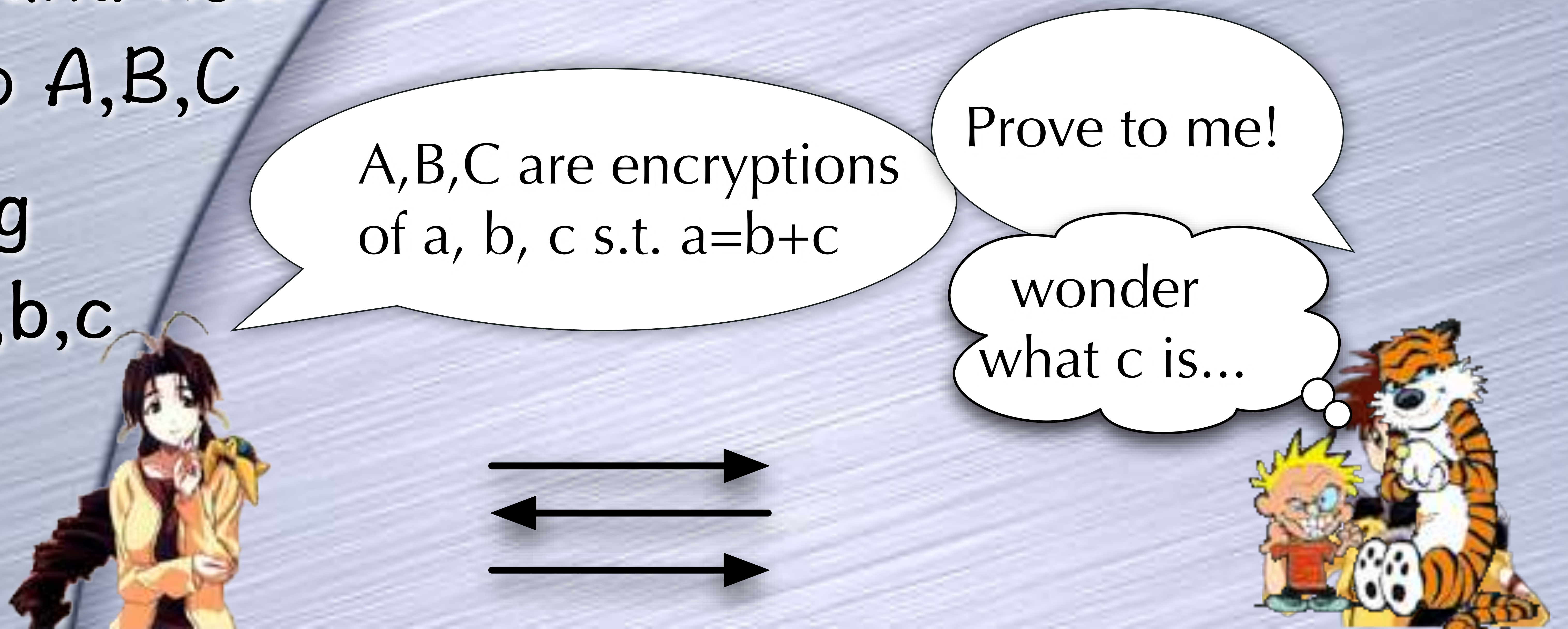
- Proving membership in an NP language L
  - x ∈ L iff ∃w R(x,w)=1 (for R in P)
    - e.g. Graph Isomorphism
- IP protocol:
  - prover just sends w
- But what if prover doesn't want to reveal w?

NP is the class of languages which have <u>non-interactive</u> and <u>deterministic</u> proof-systems



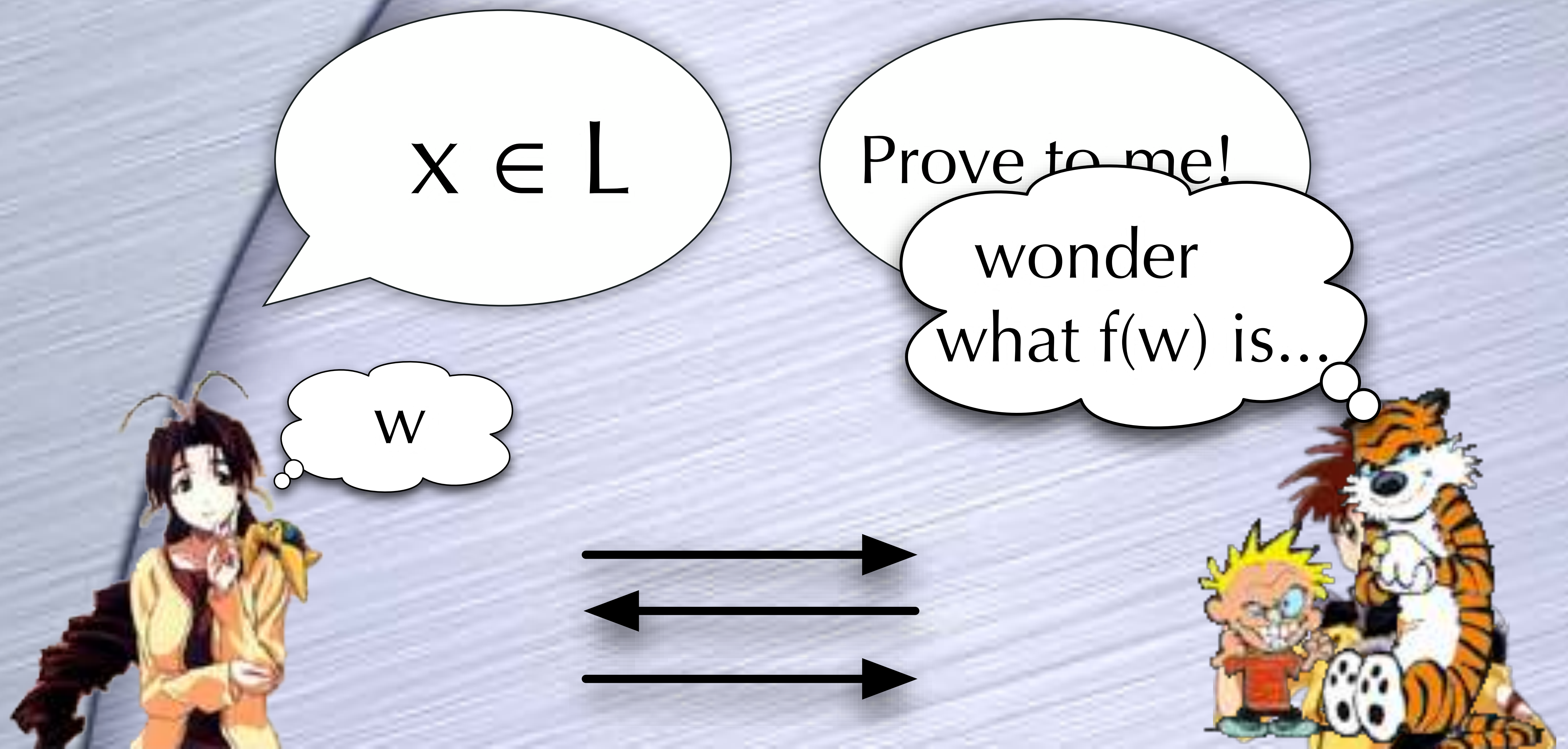
Zero-Knowledge Proofs

- In cryptographic settings, often need to be able to verify various claims
  - e.g., 3 encryptions A,B,C are of values a,b,c s.t. a=b+c
  - Option 1: reveal a,b,c and how they get encrypted into A,B,C
  - Prove without revealing anything at all about a,b,c except that a=b+c ?



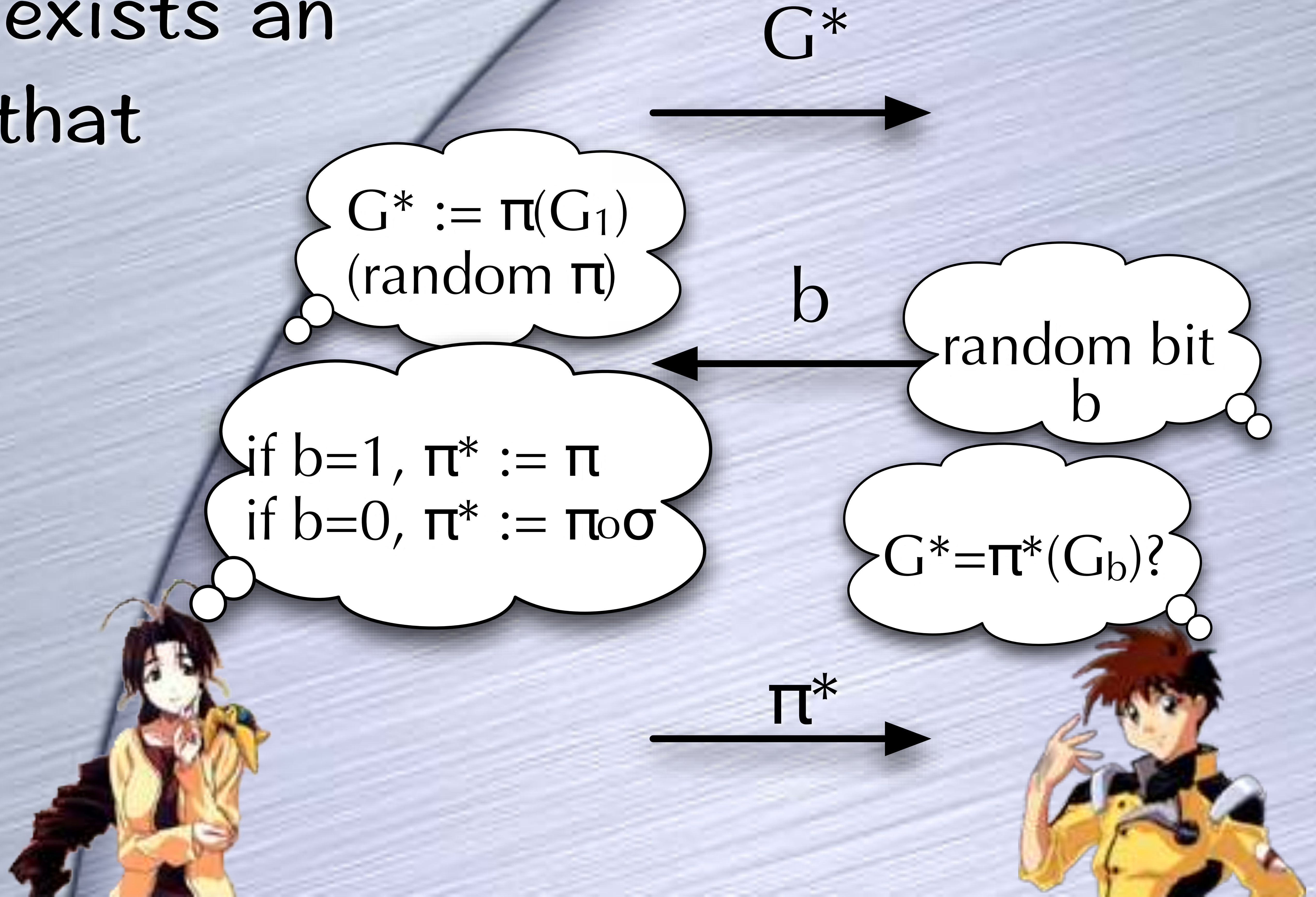
Zero-Knowledge Proofs

- Verifier should not gain any knowledge from the honest prover
  - except whether x is in L
- How to formalize this?
  - Simulation!



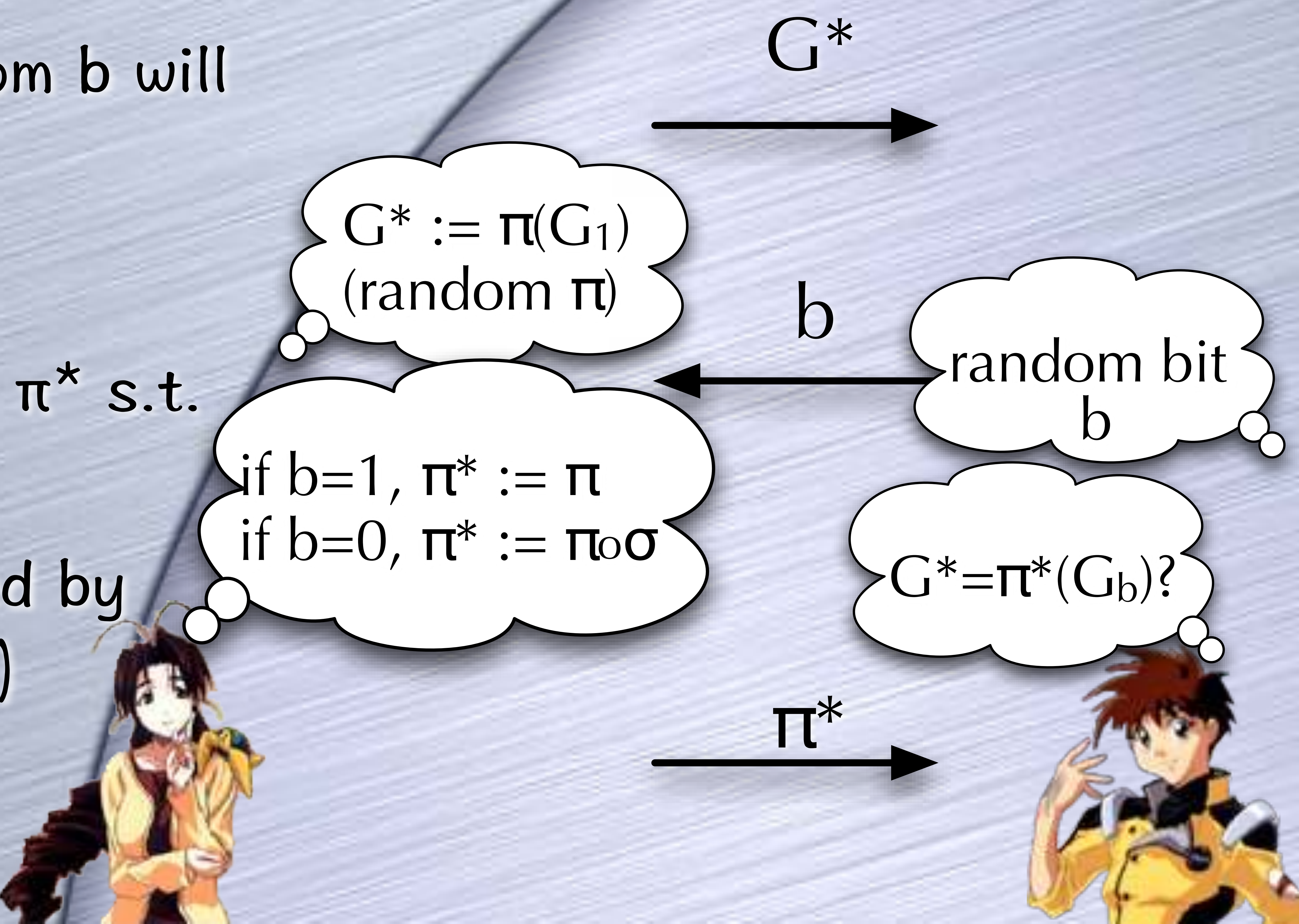
An Example

- Graph Isomorphism
  - (G_0, G_1) in L iff there exists an isomorphism σ such that $\sigma(G_0)=G_1$
- IP protocol: send σ
- ZK protocol?



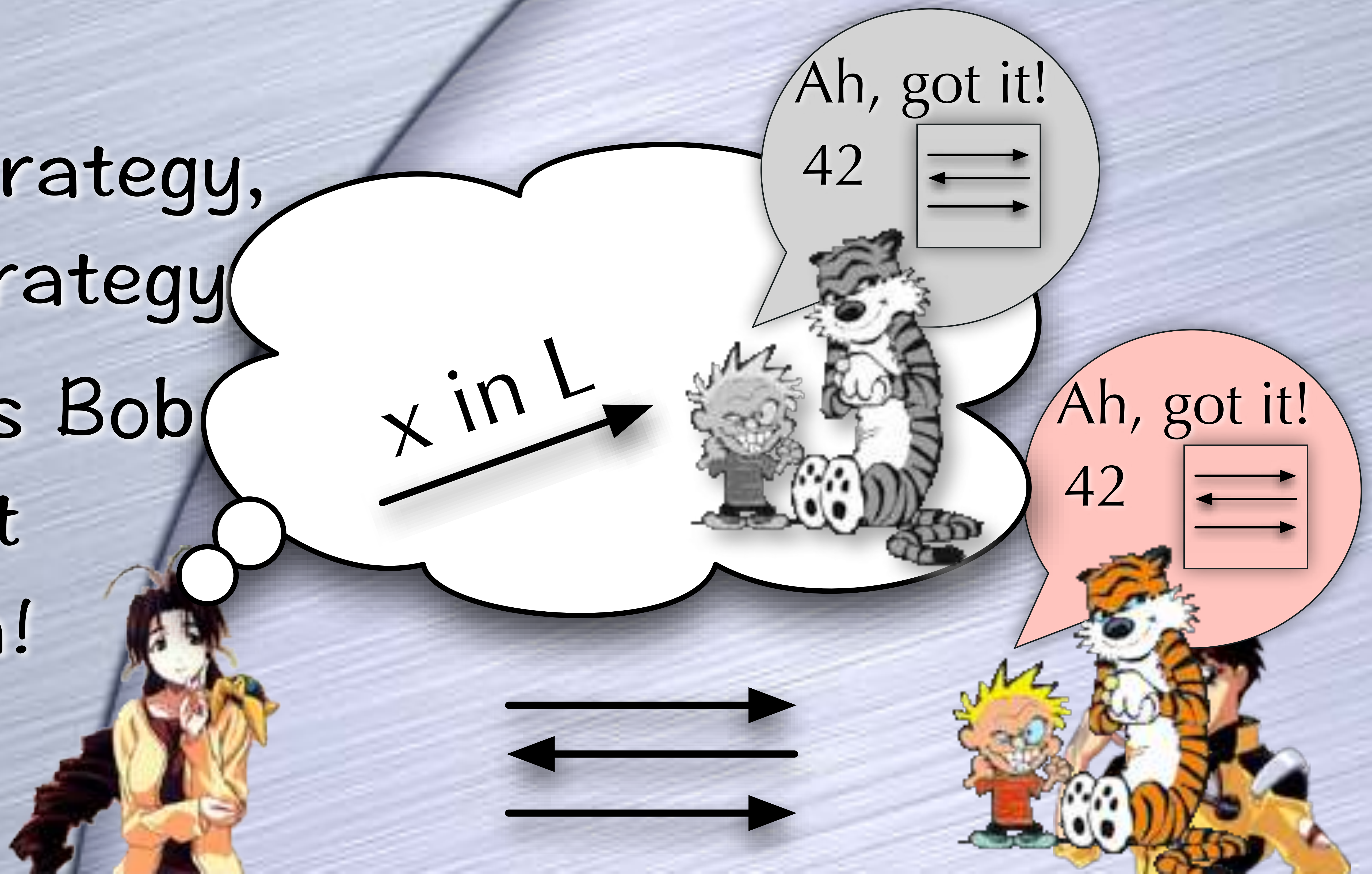
An Example

- Why is this convincing?
  - If prover can answer both b's for the same G* then $G_0 \sim G_1$
  - Otherwise, testing on a random b will leave prover stuck w.p. 1/2
- Why ZK?
  - Verifier's view: random b and π* s.t. $G^* = \pi^*(G_b)$
  - Which he could have generated by himself (whether $G_0 \sim G_1$ or not)



Zero-Knowledge Proofs

- Interactive Proof: Complete and Sound
- And has ZK Property:
  - Verifier's view could have been "simulated"
  - For every adversarial strategy, there is a simulation strategy
- Even though the view gives Bob no additional knowledge, it convinces him of the claim!



The Legend of William Tell

A Side Story

***Bob**: William Tell is a great marksman!*

***Charlie**: How do you know?*

***Bob**: I just saw him shoot an apple placed on his son's head! See this!*

***Charlie**: That apple convinced you? Anyone could have made it up!*

***Bob**: But I saw him shoot it...*

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Bob: G_0 and G_1 are isomorphic!

Charlie: How do you know?

Bob: Alice just proved it to me! See this:

$$G^*, b, \pi^* \text{ s.t. } G^*=\pi^*(G_b)$$

Charlie: That convinced you? Anyone could have made it up!

Bob: But I picked b at random and she had no trouble answering me...

Simulation

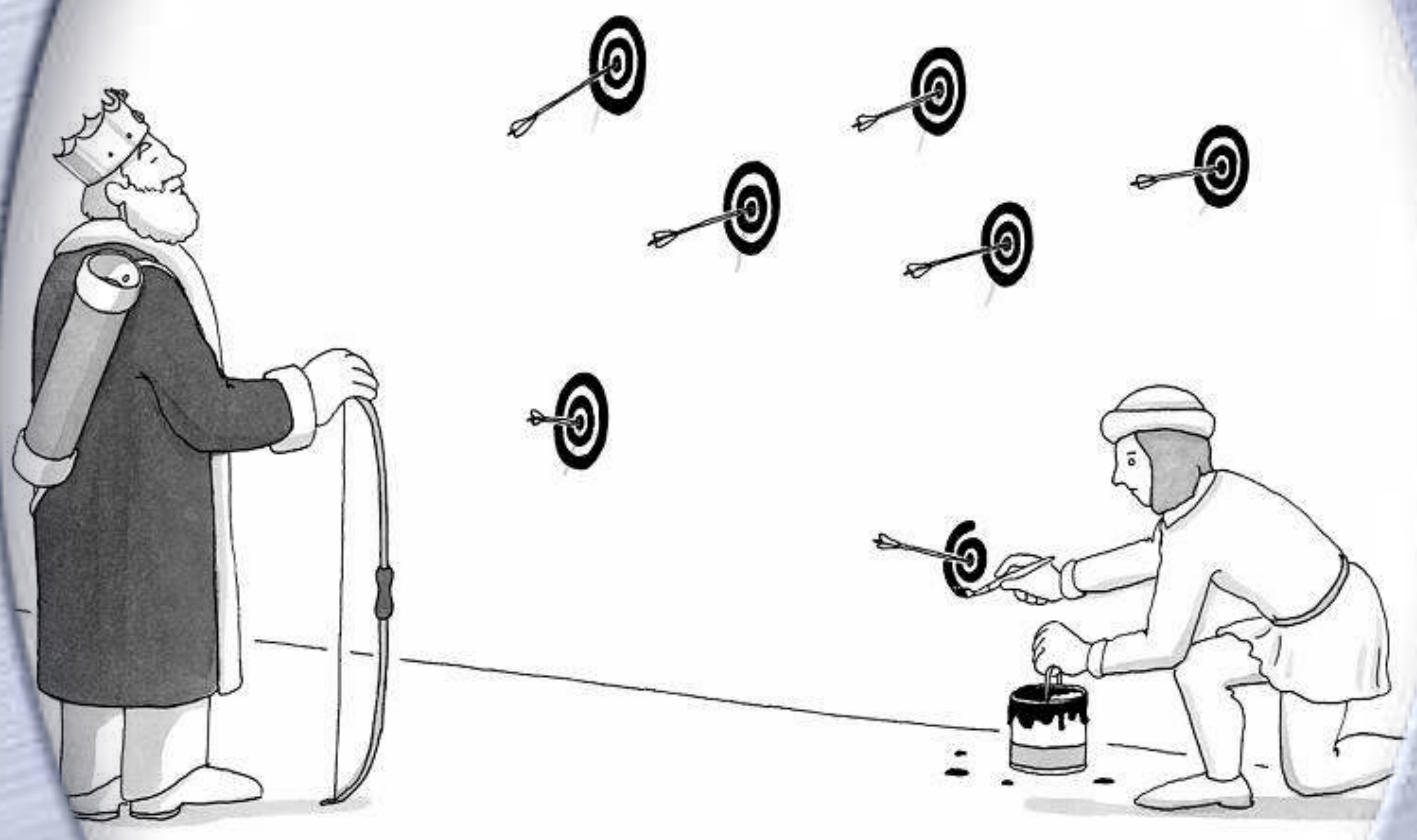
Another Analogy

- Shooting arrows at targets drawn randomly on a wall

  vs.

- Drawing targets around arrows shot randomly on to the wall

- Both produce identical views, but one of them is convincing of marksmanship



by CHARLIE HANKIN New Yorker Cartoons

Commitment

- **Commitment** is a useful tool in many ZK proofs
  - Committing to a value: Alice puts the message in a box, locks it, and sends the locked box to Bob, who learns nothing about the message
  - Revealing a value: Alice sends the key to Bob. At this point she can't influence the message that Bob will get on opening the box.
- Implementation in the Random Oracle Model: Commit(x) = H(x,r) where r is a long enough random string, and H is a random hash function (available as an oracle) with a long enough output. To reveal, send (x,r).
  - ⚠️ Recall: ROM is a heuristic model: Can do provably impossible tasks in this model! Commitment protocols exist in the standard model too.

A ZK Proof for Graph Colourability

- To prove that nodes of a graph can be coloured with at most 3 colours, so that adjacent nodes have different colours
- Uses a commitment protocol as a subroutine
- At least 1/#edges probability of catching a wrong proof
- Repeat many times with independent colour permutations
- Graph 3-colourability is an NP-complete problem
  - A ZK proof system for any NP language L:
    $x \in L$ iff $G_x \in 3COL$
    So prove $G_x \in 3COL$ instead