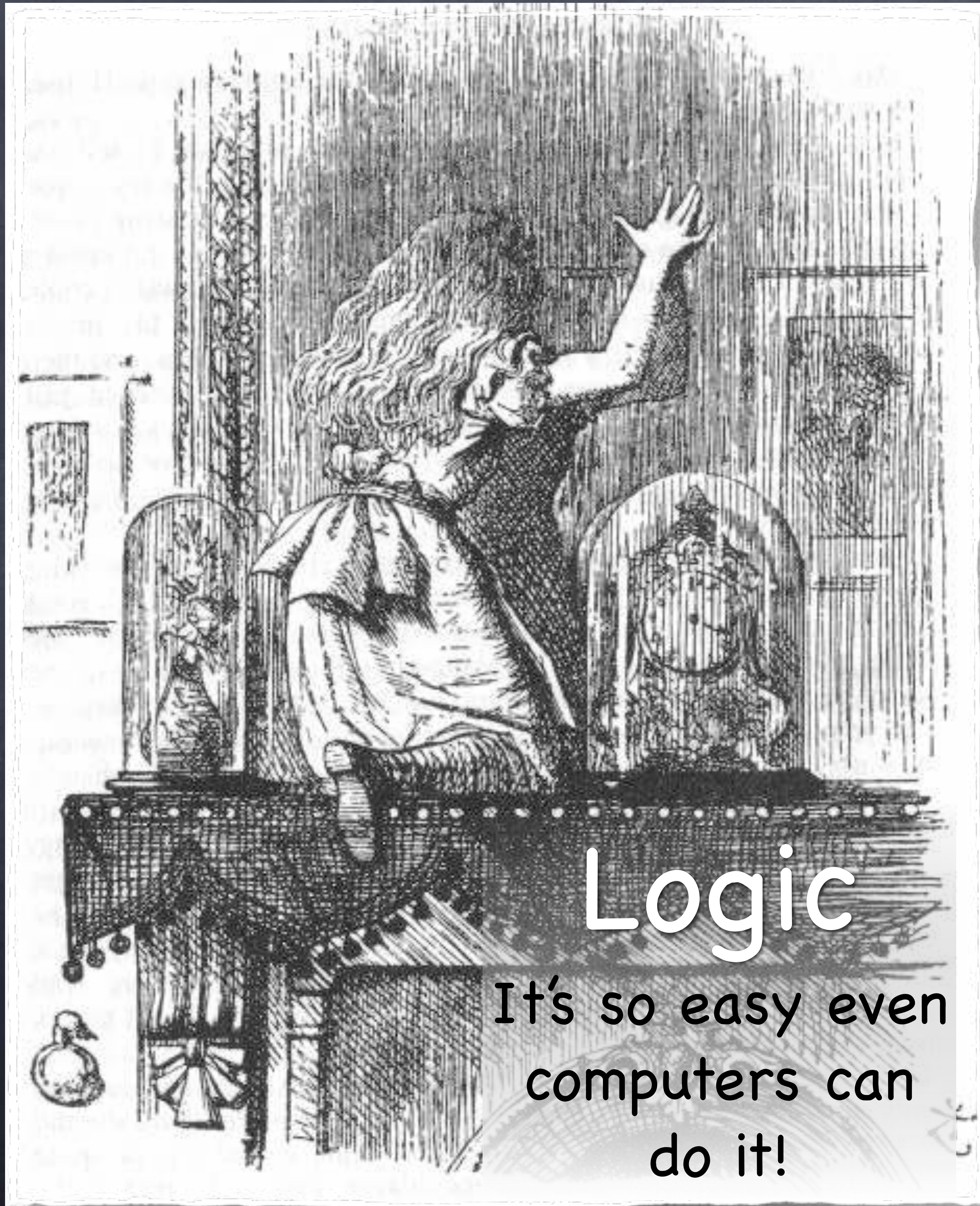


Logic

It's so easy even computers can do it!

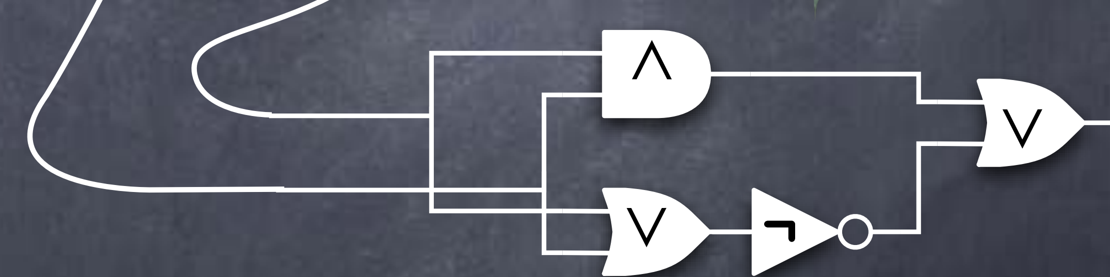
Through the Looking Glass

SAFE test

- Can you see the “Welcome” quiz on SAFE?
  - A. Yes :-)
  - B. No :-/

Story So Far

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE



- Propositions from predicates
- Propositions by applying formulas to propositions
- Propositions by applying quantifiers to predicates
  - $\forall x\ P(x)$, $\exists x\ P(x)$
- Today: Manipulating propositions

Question

- $p \rightarrow q$ is equivalent to

  A. $p \vee q$
  B. $p \wedge q$
  C. $\neg p \vee q$
  D. $\neg p \wedge q$
  E. $\neg p \vee \neg q$

Question

- Everyone who flies is winged

A. $\forall x$ Flies(x) $\vee$ Winged(x)
B. $\forall x$ Flies(x) $\wedge$ Winged(x)
C. $\forall x$ Flies(x) $\wedge$ $\neg$Winged(x)
D. $\forall x$ $\neg$Flies(x) $\vee$ Winged(x)
E. $\forall x$ $\neg$Flies(x) $\wedge$ Winged(x)

$\forall x$ Flies(x)$\rightarrow$Winged(x)

Manipulating Propositions
(Exercise)

- Conjunction and disjunction with T and F

$T \wedge q \equiv q$ | $F \vee q \equiv q$

$F \wedge q \equiv F$ | $T \vee q \equiv T$

- Implication involving T and F

$T \rightarrow q \equiv q$ | $F \rightarrow q \equiv T$

$q \rightarrow F \equiv \neg q$ | $q \rightarrow T \equiv T$

- Implication involving negation

$q \rightarrow \neg q \equiv \neg q$ | $\neg q \rightarrow q \equiv q$

- **Contrapositive**

$p \rightarrow q \equiv (\neg q) \rightarrow (\neg p)$

- **Distributive Property**

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

The Looking Glass

- A mirror which shows the negation of every proposition
- Reflection changes T & F to F & T (resp.)
  - ∨ & ∧ are reflected as ∧ & ∨ (resp.)

Flies(Alice) | ¬ Flies(Alice)

Flies(Alice) ∨ Flies(J'wock) is True

∨	T	F
T	T	T
F	T	F

∧	F	T
F	F	F
T	F	T

¬Flies(Alice) ? ¬Flies(J'wock) is False

?	F	T
F	F	F
T	F	T

∨	T	F
T	T	T
F	T	F

The Looking Glass

- A mirror which shows the negation of every ~~proposition~~ wire
- Reflection changes T & F to F & T (resp.)
  - $\vee$ & $\wedge$ are reflected as $\wedge$ & $\vee$ (resp.)

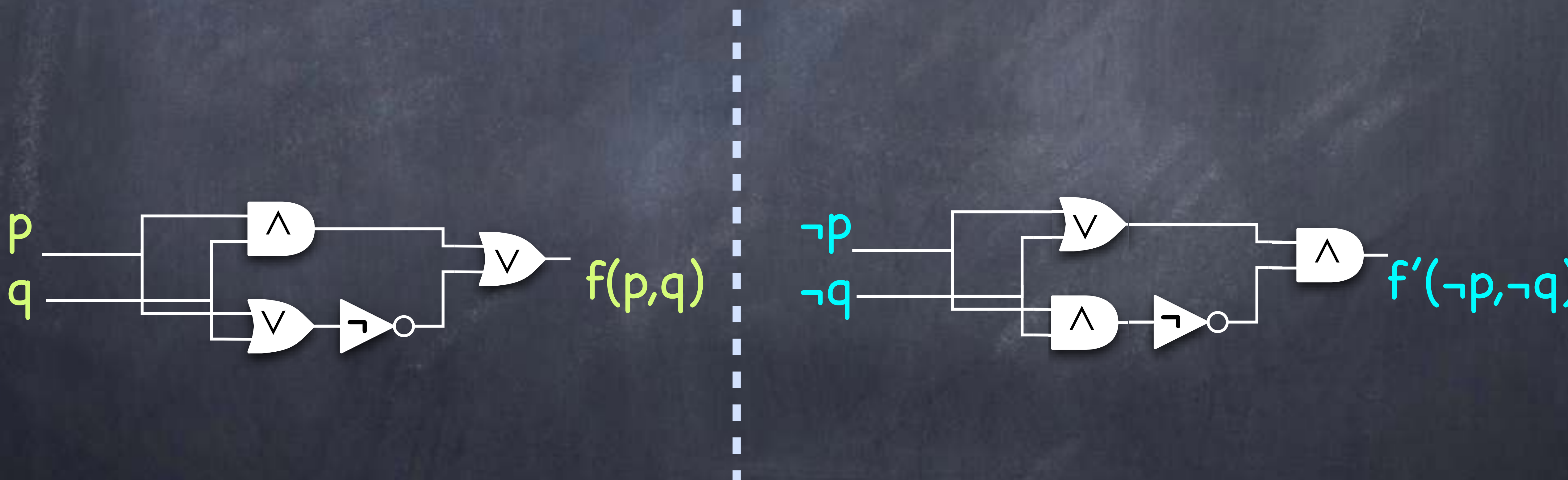
De Morgan's Law

$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

The Looking Glass

- A mirror which shows the negation of every ~~proposition~~ wire
- Reflection changes T & F to F & T (resp.)
  - $\vee$ & $\wedge$ are reflected as $\wedge$ & $\vee$ (resp.)



$$\neg\, f(p,q) \equiv f'(\neg p, \neg q)$$

Quantified Propositions

(First-Order) Predicate Calculus

x	Winged(x)	Flies(x)	Pink(x)	¬Winged(x)
Alice	FALSE	FALSE	FALSE	TRUE
Jabberwock	TRUE	TRUE	FALSE	FALSE
Flamingo	TRUE	TRUE	TRUE	FALSE

- $\forall x$ Winged(x) is False
  - Not everyone is winged
  - Same as saying, there is someone who is not winged
    - i.e., $\exists x\ \neg$Winged(x) is True

$$\neg(W(a) \wedge W(j) \wedge W(f))$$

$$\equiv$$

$$\neg W(a) \vee \neg W(j) \vee \neg W(f)$$

- $\neg\ (\ \forall x$ Winged(x)) $\equiv\ \exists x\ \neg$Winged(x)

The Looking Glass

- Reflection changes T & F to F & T (resp.)
  - ∨ & ∧ are reflected as ∧ & ∨ (resp.)
  - ∀ & ∃ are reflected as ∃ & ∀ (resp.)

p, q → ∧ → p∧q	¬p, ¬q → ∨ → ¬p ∨ ¬q
p, q → ∨ → p∨q	¬p, ¬q → ∧ → ¬p ∧ ¬q
∀x Pred(x)	∃x ¬Pred(x)
∃x Pred(x)	∀x ¬Pred(x)

Predicates, again

- A predicate can be defined over any number of elements from the domain
  - e.g., Likes(x,y): “x likes y”

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

Two quantifiers

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

- And we can quantify all the variables of a predicate
- e.g. $\forall x,y$ Likes(x,y)
  - Everyone likes everyone
  - False!

Two quantifiers

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

- ∀x ∃y Likes(x,y)
  - Everyone likes someone (True)
- ∃y ∀x Likes(x,y)
  - Someone is liked by everyone (False)

Order of quantifiers is important!

Two quantifiers

x	y	Likes(x,y)	∃y Likes(x,y) i.e., LikesSomeone(x)
Alice	Alice	TRUE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	
Jabberwock	Alice	FALSE	TRUE
	Jabberwock	TRUE	
	Flamingo	FALSE	
Flamingo	Alice	FALSE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	

- ∀x ∃y Likes(x,y)
    - Everyone likes someone
    - ∀x LikesSomeone(x)
    - True

Two quantifiers

x	y	Likes(x,y)	∃y Likes(x,y) i.e., LikesSomeone(x)
Alice	Alice	TRUE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	
Jabberwock	Alice	FALSE	TRUE
	Jabberwock	TRUE	
	Flamingo	FALSE	
Flamingo	Alice	FALSE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	

- $\forall x\ \exists y\ Likes(x,y)$
  - Everyone likes someone
  - $\forall x\ LikesSomeone(x)$
  - True
- $\exists x\ \neg(\ \exists y\ Likes(x,y)\ )$

Two quantifiers

x	y	Likes(x,y)	∃y Likes(x,y) i.e., LikesSomeone(x)
Alice	Alice	TRUE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	
Jabberwock	Alice	FALSE	TRUE
	Jabberwock	TRUE	
	Flamingo	FALSE	
Flamingo	Alice	FALSE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	

- ∀x ∃y Likes(x,y)
  - Everyone likes someone
  - ∀x LikesSomeone(x)
  - True
- ∃x ∀y ¬Likes(x,y)
  - Someone doesn't like anyone
  - ∃x DoesntLikeAnyone(x)
  - False

Two quantifiers

x	y	Likes(x,y)
Alice	Alice	TRUE
	Jabberwock	FALSE
	Flamingo	TRUE
Jabberwock	Alice	FALSE
	Jabberwock	TRUE
	Flamingo	FALSE
Flamingo	Alice	FALSE
	Jabberwock	FALSE
	Flamingo	TRUE

- $\exists y\ \underline{\forall x\ Likes(x,y)}$

Two quantifiers

x	y	Likes(x,y)	∀x Likes(x,y) i.e., EveryoneLikes(y)
Alice	Alice	TRUE	FALSE
Jabberwock		FALSE	
Flamingo		FALSE	
Alice	Jabberwock	FALSE	FALSE
Jabberwock		TRUE	
Flamingo		FALSE	
Alice	Flamingo	TRUE	FALSE
Jabberwock		FALSE	
Flamingo		TRUE	

- $\exists y$ <u>$\forall x$ Likes(x,y)</u>
  - Someone is liked by everyone
  - False

- $\forall y$ <u>$\exists x$ ¬Likes(x,y)</u>
  - Everyone is disliked by someone
  - True

Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$ for all pairs (x,y), P(x,y) holds
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$ for some pair (x,y), P(x,y) holds
- $\forall x\ P(x) \lor R \equiv (\forall x\ P(x)\ ) \lor R$ (where R is independent of x)

- **Scope** of x extends to the end: $\forall x\ (\underline{P(x) \lor R})$
- i.e., if domain is $\{a_1,\ldots,a_N\}$ $(\underline{P(a_1)\lor R}) \land \ldots \land (\underline{P(a_N)\lor R})$

- R evaluates to True or False (indep of x)
- When R is True, both equivalent (to True)
- Also, when R is False, both equivalent
- Hence both equivalent

Moving the Quantifiers

- $\forall x \ \forall y \ P(x,y) \equiv \forall y \ \forall x \ P(x,y)$ for all pairs (x,y), P(x,y) holds
- $\exists x \ \exists y \ P(x,y) \equiv \exists y \ \exists x \ P(x,y)$ for some pair (x,y), P(x,y) holds
- $\forall x \ P(x) \lor R \equiv (\forall x \ P(x)\ ) \lor R$ (where R is independent of x)

  $\forall x \ P(x) \land R \equiv (\forall x \ P(x)\ ) \land R$

  $\exists x \ P(x) \lor R \equiv (\exists x \ P(x)\ ) \lor R$

  $\exists x \ P(x) \land R \equiv (\exists x \ P(x)\ ) \land R$

- $\forall x \ \underline{R \rightarrow P(x)} \equiv R \rightarrow (\forall x \ P(x)\ )$

  $\exists x \ \underline{R \rightarrow P(x)} \equiv R \rightarrow (\exists x \ P(x)\ )$

Question

- $\forall x\ \underline{P(x) \rightarrow R}$ is equivalent to:

  A. $(\forall x\ P(x)) \rightarrow R$
  B. $(\exists x\ P(x)) \rightarrow R$
  C. $(\forall x\ P(x)) \vee R$
  D. $(\exists x\ P(x)) \vee R$
  E. $(\forall x\ P(x)) \wedge R$

$$\forall x\ \underline{\neg P(x) \vee R}$$
$$\equiv (\forall x\ \underline{\neg P(x)}) \vee R$$
$$\equiv \neg\ (\exists x\ \underline{P(x)}) \vee R$$

Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$
- When R is independent of x

$\forall x\ P(x) \vee R \equiv (\forall x\ P(x)) \vee R$	$\forall x\ P(x) \wedge R \equiv (\forall x\ P(x)) \wedge R$
$\exists x\ P(x) \vee R \equiv (\exists x\ P(x)) \vee R$	$\exists x\ P(x) \wedge R \equiv (\exists x\ P(x)) \wedge R$
$\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$	$\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$
$\forall x\ P(x) \rightarrow R \equiv (\exists x\ P(x)) \rightarrow R$	$\exists x\ P(x) \rightarrow R \equiv (\forall x\ P(x)) \rightarrow R$

- $(\forall x\ P(x)) \wedge (\forall x\ Q(x)) \equiv \forall x\ (P(x) \wedge Q(x))$
  - But $(\forall x\ P(x)) \vee (\forall x\ Q(x)) \not\equiv \forall x\ (P(x) \vee Q(x))$ — Not equivalent to!
- $(\exists x\ P(x)) \vee (\exists x\ Q(x)) \equiv \exists x\ (P(x) \vee Q(x))$
  - But $(\exists x\ P(x)) \wedge (\exists x\ Q(x)) \not\equiv \exists x\ (P(x) \wedge Q(x))$

Today

- Negating propositions (the looking glass)
  - De Morgan's law
  - When quantifiers are involved
- Multiple quantifiers
  - Order of quantifiers matters
  - Negation
- Moving quantifiers around