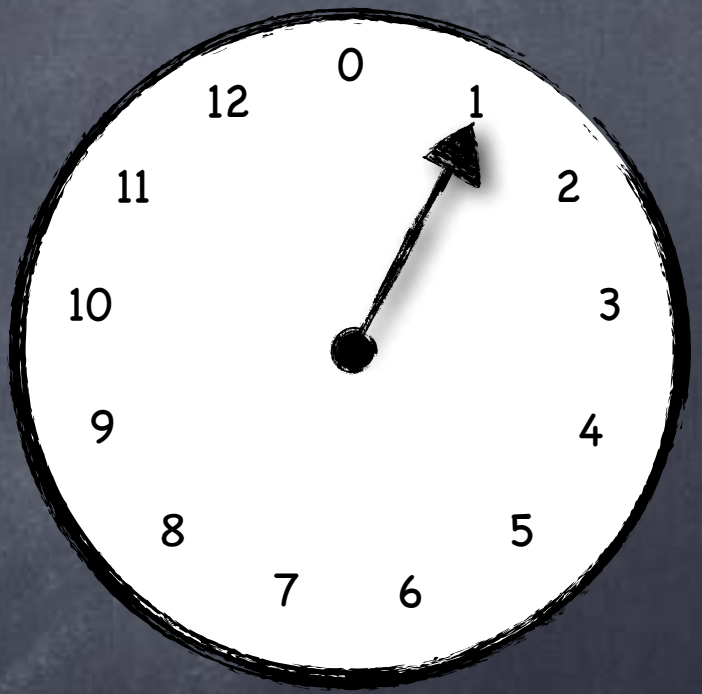
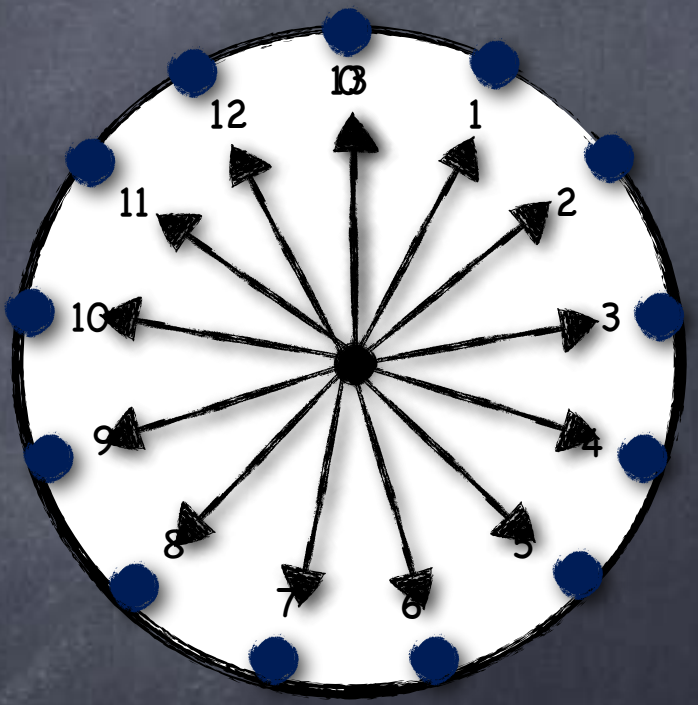
Numb3rs

Lecture 4



The Skippy Clock

- Has 13 hours on its dial!
- Needle moves two hours at a time
- Which all numbers will the needle reach?
  - Reaches all of them!
    - Because it reaches 1!



Integers: Basics

- $\mathbb{Z}$: set of all integers { ..., -2, -1, 0, 1, 2, ... }
  - Operations addition, subtraction and multiplication (and their various properties)
- Definition: For $a,b \in \mathbb{Z}$, $a|b$ (a divides b) if $\exists q \in \mathbb{Z} \;\; b = qa$
  - $a|b \equiv$ b is a multiple of a $\equiv$ a is a divisor of b
  - Multiples of a : { ..., -2a, -a, 0, a, 2a, ... }
  - Divisors of b: all a such that $a|b$
    [a.k.a. factors]

Question

- Consider the following two statements:

  (I) $\forall\ a \in \mathbb{Z},\ a \mid 0$

  (II) $\forall\ b \in \mathbb{Z},\ -1 \mid b$

  A. One of them is undefined
  B. (I) is true and (II) is false
  C. (II) is true and (I) is false
  D. Both are true
  E. Both are false

$0 = 0.a$
$b = (-b).(-1)$

Integers: Basics

- Proposition: $\forall\ a,b,c \in \mathbb{Z}$ if $a|b$, then $a|bc$

  $b = qa$
  $\Rightarrow bc = q'a$, where $q'=qc$

- Proposition: $\forall\ a,b,c \in \mathbb{Z}$ if $a|b$ and $a|c$, then $a|(b+c)$

  $b = qa \ \&\ c = q'a$
  $\Rightarrow b+c = q''a$, where $q''=q+q'$

- Proposition: $\forall\ a,b,c \in \mathbb{Z}$ if $a|b$ and $b|c$, then $a|c$

  $b = qa \ \&\ c = q'b$
  $\Rightarrow c = q''a$, where $q''=qq'$

- Proposition: $\forall\ a,b,c \in \mathbb{Z}$ if $ac|bc$ and $c \neq 0$, then $a|b$

  $bc = qac \ \&\ c \neq 0$
  $\Rightarrow b = qa$

- Proposition: $\forall\ a,b \in \mathbb{Z}$ if $a|b$ and $b \neq 0$, then $|a| \leq |b|$

  $b = qa \ \&\ b \neq 0 \Rightarrow |b| = |q| \cdot |a|$ where $|q| \geq 1$
  $\Rightarrow |b| = |a| + (|q|-1) \cdot |a| \geq |a|$

Division

For any two integers a and b, a≠0, there is a unique quotient q and remainder r (integers), such that

$$b = q \cdot a + r, \quad 0 \le r < |a|$$

- Proof of existence
  - We shall prove it for all non-negative b and positive a. Then, the other cases can be proven as follows:
    - a>0, b<0: $b = -(-b) = -(q \cdot a + r) = -(q+1)a + (a-r)$, and $0 \le a-r < a$
    - a<0, b>0: $b = q \cdot (-a) + r = -qa + r$, and $0 \le r < |a|$
    - a<0, b<0: $b = -(-b) = -(q \cdot (-a) + r) = (q+1)a + (-a-r)$, and $0 \le -a-r < |a|$
  - Fix any a>0. We use strong induction on b.
  - Base cases: $b \in [0,a)$. Then let q=0 and r=b : $b = 0.a + b$.
  - Induction step: We shall prove that for all $k \ge a$,

    (induction hypothesis): if $\forall b \in \mathbb{Z}^+$ s.t. b<k, $\exists q,r$ s.t $b=qa+r$ & $0 \le r \le a$

    (to prove): then $\exists q^*, r^*$ s.t. $k = q^* \cdot a + r^*$ & $0 \le r^* \le a$.
  - Consider $k'=k-a$. $0 \le k' < k$. By ind. hyp. $k'=q'a+r'$. Let $q^*=q'+1$, $r^*=r'$. □

Here, case r>0. If r=0, b = ±qa

Division

For any two integers a and b, $a \neq 0$, there is a unique quotient q and remainder r (integers), such that

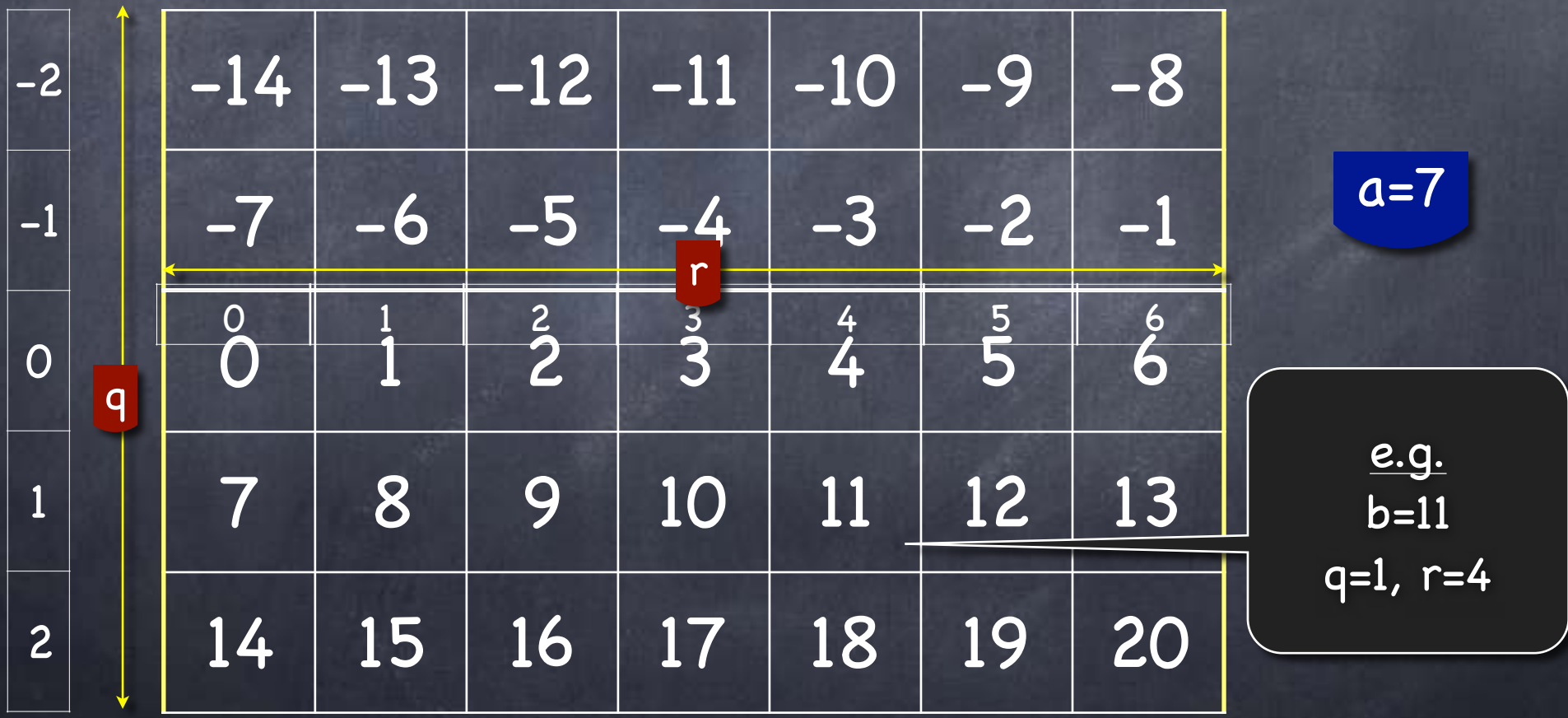
$$b = q \cdot a + r, \quad 0 \leq r < |a|$$

- Proof of existence
  - Also known as "Division Algorithm" (when you unroll the inductive argument, you get a (naïve) algorithm)
- Proof of uniqueness:
  - Claim: if $b = q_1 \cdot a + r_1 = q_2 \cdot a + r_2$, where $0 \leq r_1, r_2 < |a|$, then $q_1 = q_2$ and $r_1 = r_2$
    - Suppose, $q_1 \cdot a + r_1 = q_2 \cdot a + r_2$. Then $(r_1 - r_2) = (q_2 - q_1)a$. i.e., $a | (r_1 - r_2)$.
    - W.l.o.g, $r_1 \geq r_2$. So, $0 \leq (r_1 - r_2) < |a|$. Now, the only multiple of a in that range is 0. So $r_1 = r_2$. Then $(q_1 - q_2)a = 0$. Since $a \neq 0$, $q_1 = q_2$.

Division

For any two integers a and b, $a \neq 0$, there is a unique quotient q and remainder r (integers), such that

$$b = q \cdot a + r, \quad 0 \leq r < |a|$$



Common Factors

- Common Divisor: c is a common divisor of integers a and b if c|a and c|b. [a.k.a. common factor]

- Greatest Common Divisor (for (a,b)≠(0,0))
  gcd(a,b) = largest among common divisors of a and b

  - Well-defined: 1 is always a common factor. And, no common factor is larger than min(|a|,|b|) (unless a=b=0). So gcd(a,b) is an integer in the range [1, min(|a|,|b|)].

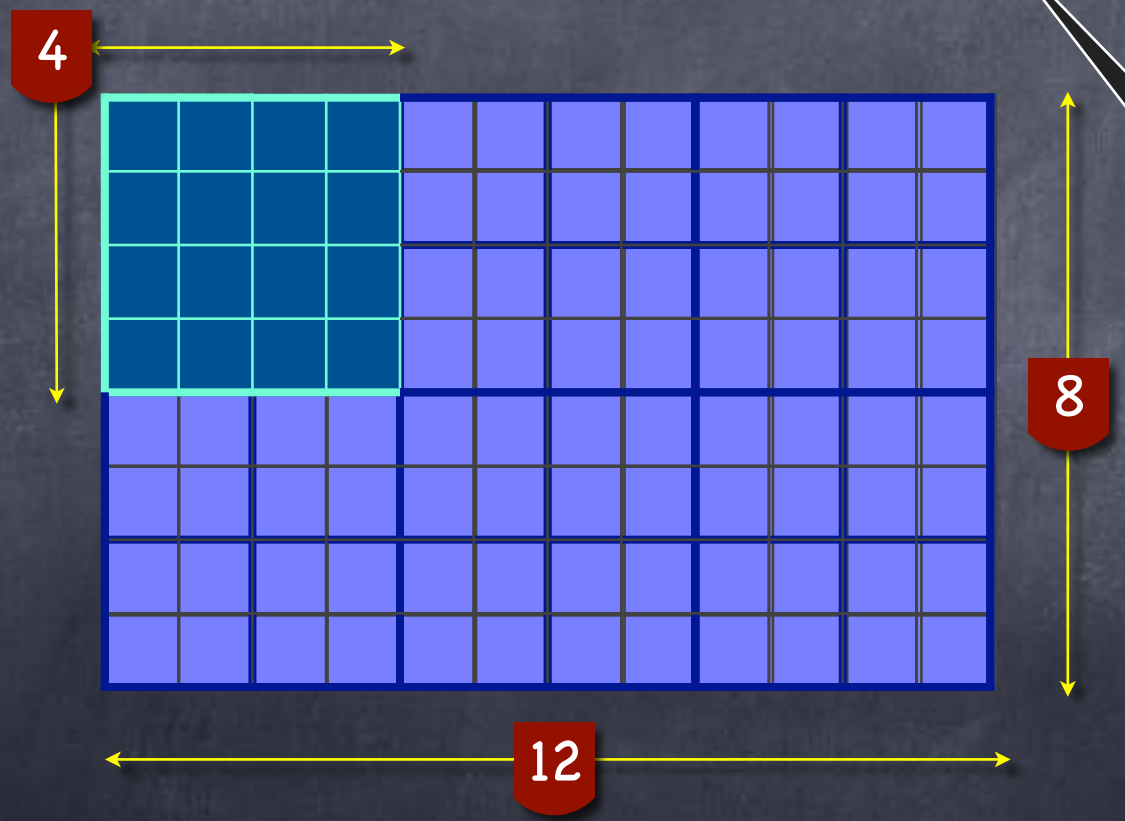
  - e.g. Divisors(12) = { ±1, ±2, ±3, ±4, ±6, ±12 }.
    Divisors(18) = { ±1, ±2, ±3, ±6, ±9, ±18 }.
    Common-divisors(12,18) = { ±1, ±2, ±3, ±6 }. gcd(12,18) = 6

  - e.g. Divisors(0) = $\mathbb{Z}$. $\forall x \neq 0$ gcd(x,0) = |x|.
    Also, $\forall x,a \in \mathbb{Z}$, $|x| \in$ Divisors(ax). If $x \neq 0$, gcd(x,ax)=|x|.

GCD as Tiling

[Here all numbers are positive integers]

- d is a common factor of a & b, iff a d x d square tile can be used to perfectly tile an a x b rectangle



GCD: largest such square tile

Common Factors

- Common Divisor: c is a common divisor of integers a and b if c|a and c|b. [a.k.a. common factor]
- Greatest Common Divisor (for $(a,b) \neq (0,0)$)
  gcd(a,b) = largest among common divisors of a and b
- $\forall a,b,n \in \mathbb{Z}$, common-divisors(a,b) = common-divisors(a,b+na)
  - i.e., $(x|a \wedge x|b) \longleftrightarrow (x|a \wedge x|b+na)$. [Verify!]
- Hence, $\forall a,b,n \in \mathbb{Z}$, gcd(a,b) = gcd(a,b+na)
  - In particular, $\forall a,b \in \mathbb{Z}$, gcd(a,b) = gcd(a,r), where $b = aq+r$ and $0 \leq r < a$

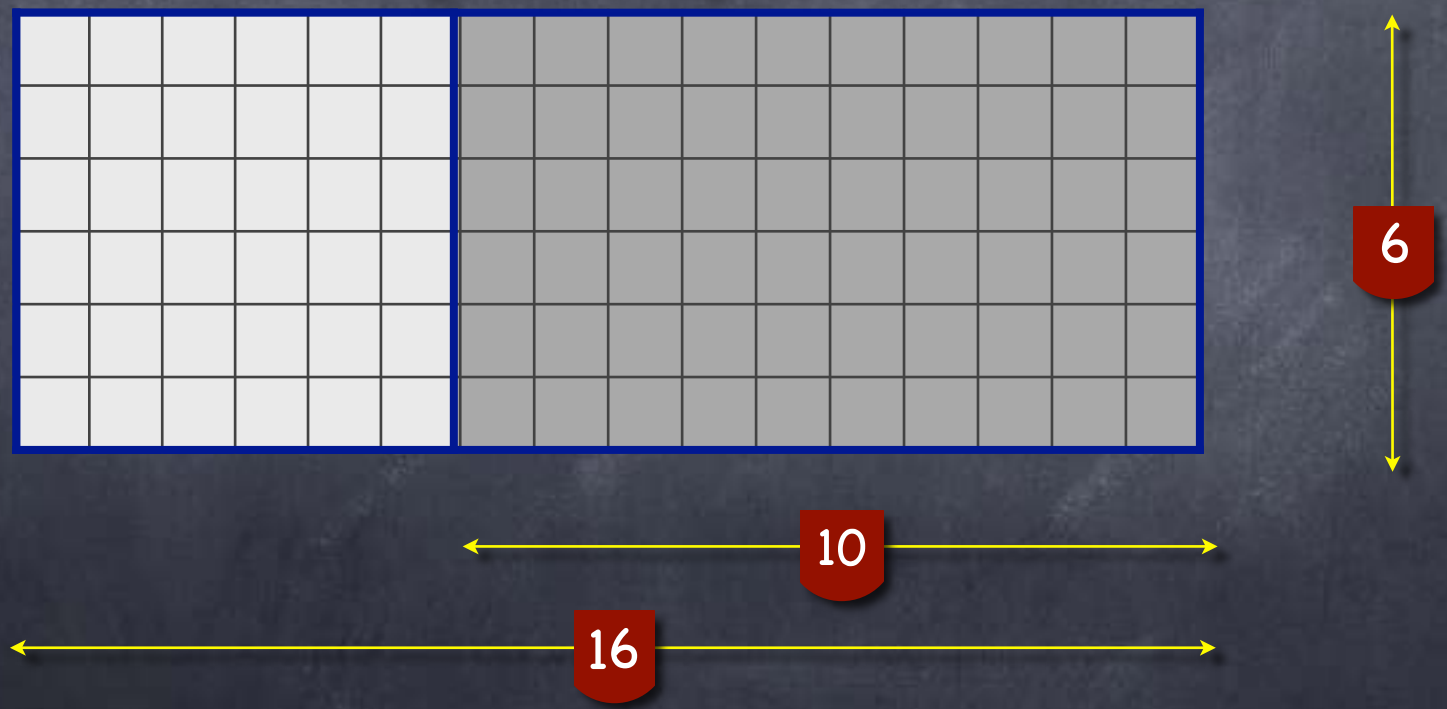
Euclid's GCD Algorithm

[Here all numbers are positive integers]

- Find the largest square perfectly tiling a x b rectangle

common-divisors(a,b) = common-divisors(a,b-a)

gcd(a,b) = gcd(a,b-a)



gcd(6,16) = gcd(6,10)

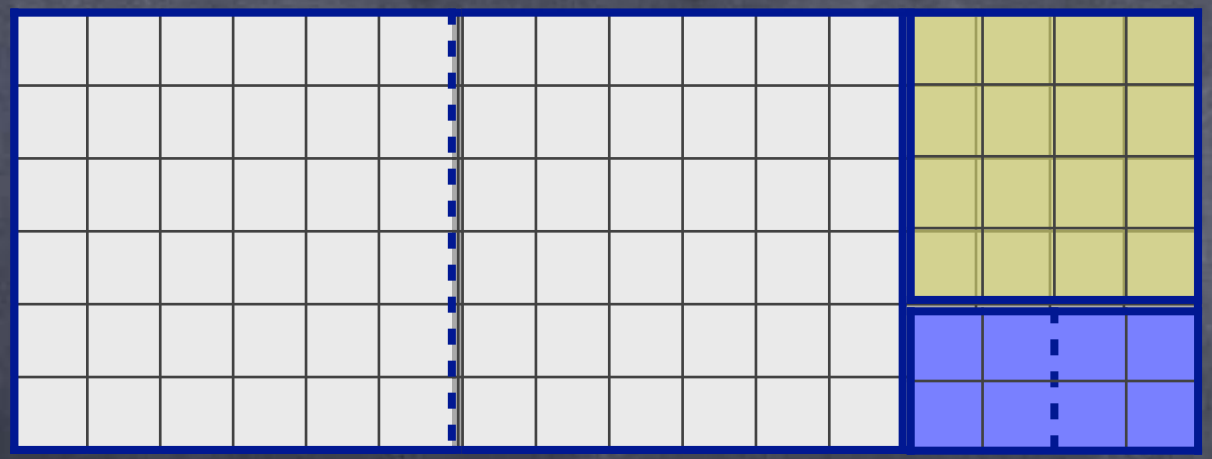
Euclid's GCD Algorithm

[Here all numbers are positive integers]

- Find the largest square perfectly tiling a x b rectangle

common-divisors(a,b) = common-divisors(a,b-qa)

gcd(a,b) = gcd(a,b-qa)



$\forall\ a,b \in \mathbb{Z}$

$\exists\ u,v \in \mathbb{Z}$

$\gcd(a,b) = u \cdot a + v \cdot b$

$3 \cdot 6 - 1 \cdot 16 = 6 - (16 - 2 \cdot 6) = 6 - 4 = 2$

$\gcd(6,16) = \gcd(6,4) = \gcd(2,4) = 2$

The Hoppy Bunny

- A bunny is sitting on an infinite number line, at position 0

- The bunny has two hops — of lengths a and b, where $a,b \in \mathbb{Z}$
  - Can hop to left or right (irrespective of the sign of a,b)
- What all points can the bunny reach?
- After u a-hops and v b-hops (u, v could be negative, indicating direction opposite a or b's sign), bunny is at $a \cdot u + b \cdot v$

- For any $a, b \in \mathbb{Z}$, let L(a,b) be the set of all integer combinations of a, b. i.e., **$L(a,b) = \{ au+bv \mid u,v \in \mathbb{Z} \}$**

The One Dimensional Lattice

- For any a, b $\in \mathbb{Z}$, let L(a,b) be the set of all integer combinations of a, b. i.e., **$L(a,b) = \{ au+bv \mid u,v \in \mathbb{Z} \}$**
- Claim: **L(a,b) consists of exactly all the multiples of gcd(a,b)**
- Proof: Note that gcd(a,b) divides every element in L(a,b). i.e., every element in L(a,b) is a multiple of gcd(a,b). We shall prove below that gcd(a,b) $\in$ L(a,b), so that all its multiples are also in L(a,b) (L(a,b) being closed under multiplication by integers).
  - By the well-ordering principle, let d be the smallest element in $L^+(a,b) \triangleq L(a,b) \cap \mathbb{Z}^+$.
  - Let d=au+bv. Let a=dq+r, where $0 \le r < d$. So, $r=a-(au+bv)q \in L(a,b)$. Since r<d, we require $r \notin L^+(a,b)$. So r=0. i.e., d|a. Similarly, d|b. That is, d is a common divisor. So, $d \le \gcd(a,b)$.
  - But $d \in L(a,b) \Rightarrow \gcd(a,b) \mid d \Rightarrow \gcd(a,b) \le d$. So $\gcd(a,b) = d \in L(a,b)$

Primes

- Definition: $p \in \mathbb{Z}$ is said to be a prime number if $p \geq 2$ and the only positive factors of p are 1 and p itself
  - 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, ...

Unique Factorisation
(Fundamental Theorem of Arithmetic):
$\forall a \in \mathbb{Z}$, if $a \geq 2$ then $\exists!$ $(p_1,\ldots,p_t, d_1,\ldots,d_t)$ s.t.

$p_1 < \ldots < p_t$ primes, $d_1,\ldots,d_t \in \mathbb{Z}^+$, and $a = p_1^{d1}\, p_2^{d2} \ldots\, p_t^{dt}$

- Recall: We already saw that prime factorisation exists (using strong induction)

- Will prove uniqueness now

Primes

- Definition: $p \in \mathbb{Z}$ is said to be a prime number if $p \geq 2$ and the only positive factors of p are 1 and p itself

- **Euclid's Lemma**

  $\forall a,b,p \in \mathbb{Z}$ s.t. p is prime $(p \mid ab) \rightarrow (p \mid a \vee p \mid b)$

- Since the only positive factors of p are 1, p, we have gcd(a,p) = 1 or gcd(a,p) = p.

- If gcd(a,p) = p, then $p \mid a$ ✓

- If gcd(a,p) = 1, $\exists u,v$ s.t. $1 = au+pv \Rightarrow b = bau + bpv \Rightarrow b \in L(ab,p)$

  But $p \mid ab$ and $p \mid p$. So $p \mid b$.

Primes

- Definition: $p \in \mathbb{Z}$ is said to be a prime number if $p \geq 2$ and the only positive factors of p are 1 and p itself

- **Euclid's Lemma**

  $\forall a,b,p \in \mathbb{Z}$ s.t. p is prime $(p \mid ab) \rightarrow (p \mid a \vee p \mid b)$

- Generalisation of Euclid's Lemma (Prove by induction):
  $\forall a_1,\ldots, a_n, p \in \mathbb{Z}$ s.t. p is prime, $(p \mid a_1 \cdots a_n) \rightarrow \exists\, i,\ p \mid a_i$

- Uniqueness of prime factorisation: Suppose z is the smallest positive integer with two distinct prime factorisations as $z = p_1 \cdots p_m = q_1 \cdots q_n$. $\max\{p_1,\ldots,p_m\} \neq \max\{q_1,\ldots,q_n\}$ (Why?). So w.l.o.g., $p_m > q_i$, i=1 to n. Now, $p_m \mid q_1 \cdots q_n \Rightarrow p_m \mid q_i$ for some i (by Lemma). This contradicts $p_m > q_i$.