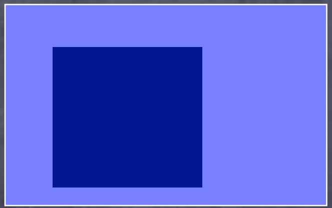
Sets and Relations

Lecture 8

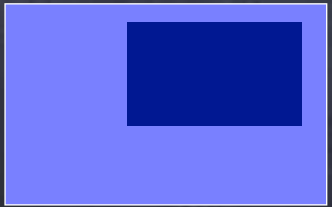
Sets: Basics

- Unordered collection of "elements"
  - e.g.: $\mathbb{Z}$, $\mathbb{R}$ (infinite sets), $\emptyset$ (empty set), {1, 2, 5}, …
- Will always be given an implicit or explicit universe (universal set) from which the elements come
  - (Aside: In developing foundations of mathematics, often one starts from "scratch", using only set theory to create the elements themselves)
- Set membership: e.g. $0.5 \in \mathbb{R}$, $0.5 \notin \mathbb{Z}$, $\emptyset \notin \mathbb{Z}$
- Set inclusion: e.g. $\mathbb{Z}, \subseteq \mathbb{R}$, $\emptyset \subseteq \mathbb{Z}$
- Set operations: complement, union, intersection, difference

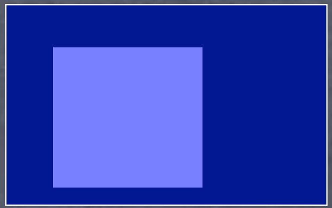
Set Operations



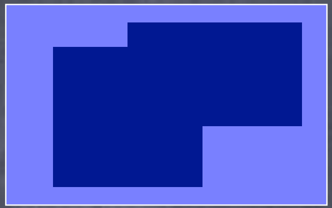
S



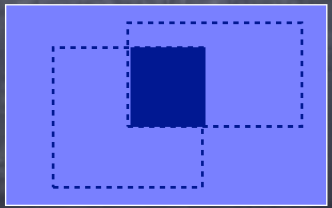
T



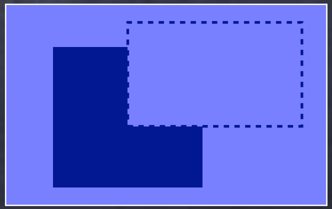
$\overline{S}$



$S \cup T$



$S \cap T$



$S - T$

Sets as Predicates

x	Winged(x)	Flies(x)	Pink(x)	inClub(x)
Alice	FALSE	FALSE	FALSE	TRUE
Jabberwock	TRUE	TRUE	FALSE	FALSE
Flamingo	TRUE	TRUE	TRUE	TRUE

- Given predicate can define the set of elements for which it holds
  - WingedSet = { x | Winged(x) } = {J'wock, Flamingo}
  - FliesSet = { x | Flies(x) } = {J'wock, Flamingo}
  - PinkSet = { x | Pink(x) } = {Flamingo}

- Given set, can define a corresponding predicate too
  e.g. given set Club = {Alice, Flamingo}. Then, define predicate
  inClub(x) s.t. inClub(x) = True iff $x \in$ Club

Set Operations

Unary operator

Binary operators

Associative

S **complement** T

Symbol: $\overline{S}$

$in\overline{S}(x) \equiv \neg inS(x)$

S **union** T

Symbol: $S \cup T$

$inS{\cup}T(x)$

$\equiv inS(x) \vee inT(x)$

S **intersection** T

Symbol: $S \cap T$

$inS{\cap}T(x)$

$\equiv inS(x) \wedge inT(x)$

S **difference** T

Symbol: $S - T$

$inS{-}T(x)$

$\equiv inS(x) \wedge \neg inT(x)$

$\equiv inS(x) \nrightarrow inT(x))$

$S-T = S \cap \overline{T}$

S **symmetric diff.** T

Symbol: $S \,\Delta\, T$

$inS\Delta T(x)$

$\equiv inS(x) \oplus inT(x)$

Note: Notation inS(x) used only to explicate the connection with predicate logic. Always write $x \in S$ instead.

De Morgan's Laws

- $\overline{S \cup T} = \bar{S} \cap \bar{T}$
  - $x \in \overline{S \cup T} \equiv \neg(x \in S \cup T)$

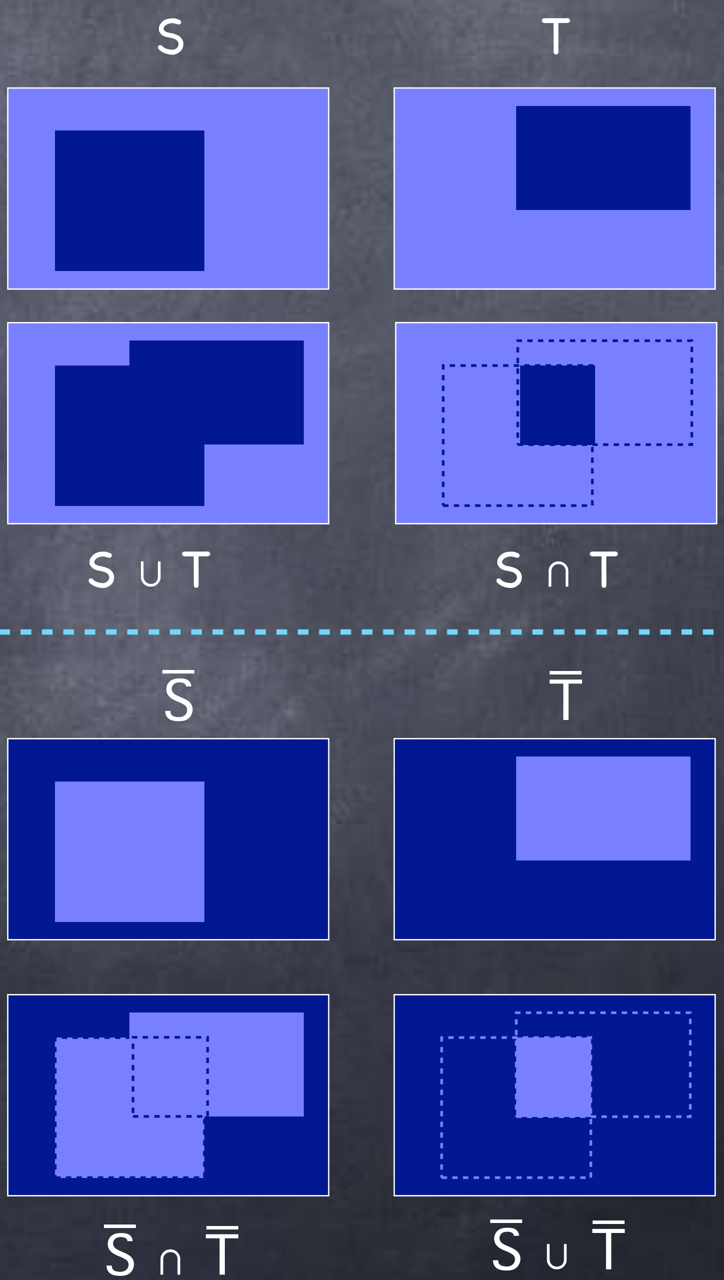
    $\equiv \neg(x \in S \vee x \in T) \equiv \neg(x \in S) \wedge \neg(x \in T)$

    $\equiv x \in \bar{S} \wedge x \in \bar{T} \equiv x \in \bar{S} \cap \bar{T}$

- $\overline{S \cap T} = \bar{S} \cup \bar{T}$
  - $x \in \overline{S \cap T} \equiv \neg(x \in S \cap T)$

    $\equiv \neg(x \in S \wedge x \in T) \equiv \neg(x \in S) \vee \neg(x \in T)$

    $\equiv x \in \bar{S} \vee x \in \bar{T} \equiv x \in \bar{S} \cup \bar{T}$



Distributivity

- $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$
    - $x \in R \cap (S \cup T) \equiv$
      $\equiv x \in R \wedge (x \in S \vee x \in T) \equiv (x \in R \wedge x \in S) \vee (x \in R \wedge x \in T)$
      $\equiv x \in (R \cap S) \cup (R \cap T)$
- $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
    - $x \in R \cup (S \cap T) \equiv$
      $\equiv x \in R \vee (x \in S \wedge x \in T) \equiv (x \in R \vee x \in S) \wedge (x \in R \vee x \in T)$
      $\equiv x \in (R \cup S) \cap (R \cup T)$

Set Inclusion

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- PinkSet $\subseteq$ FliesSet = WingedSet

- S $\subseteq$ T same as the proposition $\forall x \;\; x \in S \rightarrow x \in T$
- S $\supseteq$ T same as the proposition $\forall x \;\; x \in S \leftarrow x \in T$
- S = T same as the proposition $\forall x \;\; x \in S \leftrightarrow x \in T$

Set Inclusion

- $S \subseteq T$ same as the proposition $\underline{\forall x \;\; x \in S \rightarrow x \in T}$
- If $S = \emptyset$, and T any arbitrary set, $S \subseteq T$
  - $\forall x$, vacuously we have $x \in S \rightarrow x \in T$

If no such x, already done

- If $S \subseteq T$ and $T \subseteq R$, then $S \subseteq R$
  - Consider arbitrary $x \in S$. Since $S \subseteq T$, $x \in T$. Then since $T \subseteq R$, $x \in R$.
- $S \subseteq T \longleftrightarrow \overline{T} \subseteq \overline{S}$
  - $\forall x \;\; \underline{x \in S \rightarrow x \in T} \;\; \equiv \;\; \forall x \;\; \underline{x \notin T \rightarrow x \notin S}$ (contrapositive)

$$\equiv \;\; \forall x \;\; \underline{x \in \overline{T} \rightarrow x \in \overline{S}}$$

Proving Set Equality

- To prove $S = T$, show $S \subseteq T$ and $T \subseteq S$
- e.g., $L(a,b) = \{ x : \exists u,v \in \mathbb{Z} \;\; x=au+bv \}$

  $M(a,b) = \{ x : (\gcd(a,b) \mid x) \}$
- Recall Claim: $L(a,b) = M(a,b)$
- Proof in two parts:
  - $L(a,b) \subseteq M(a,b)$: i.e., $\forall x \in \mathbb{Z} \;\; x \in L(a,b) \rightarrow x \in M(a,b)$
  - $M(a,b) \subseteq L(a,b)$: i.e., $\forall x \in \mathbb{Z} \;\; x \in M(a,b) \rightarrow x \in L(a,b)$

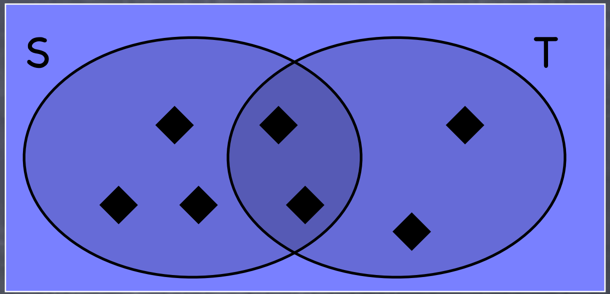
Let $x=au+bv$. $g|a$, $g|b \Rightarrow g|x$

First show that $g \in L(a,b)$ (as the smallest +ve element in $L(a,b)$)

Let $x=ng$. But $g=au+bv \Rightarrow x=au'+bv'$

Inclusion-Exclusion

- $|S| + |T|$ counts every element that is in S or in T
  - But it double counts the number of elements that are in both: i.e., elements in $S \cap T$

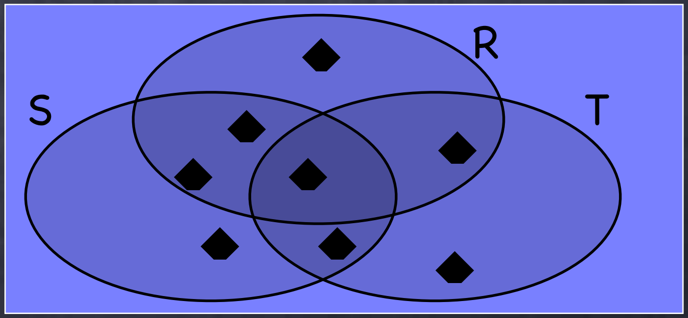


- So, $|S|+|T| = |S \cup T| + |S \cap T|$
- Or, $|S \cup T| = |S| + |T| - |S \cap T|$
- $|R \cup S \cup T| = |R|+|S|+|T| - |R \cap S| - |S \cap T| - |T \cap R| + |R \cap S \cap T|$
  - $|R \cup S \cup T| = |R| + |S \cup T| - |R \cap (S \cup T)|$

$$= |R| + |S \cup T| - |(R \cap S) \cup (R \cap T)|$$

$$= |R| + |S| + |T| - |S \cap T|$$

$$-(|R \cap S| + |R \cap T| - |R \cap S \cap T|)$$



Cartesian Product

- $S \times T = \{ (s,t) \mid s \in S \text{ and } t \in T \}$
  - $(S = \emptyset \vee T = \emptyset) \leftrightarrow S \times T = \emptyset$
  - $|S \times T| = |S| \cdot |T|$
- $R \times S \times T = \{ (r,s,t) \mid r \in R, s \in S \text{ and } t \in T \}$
  - Not the same as $(R \times S) \times T$ (but "essentially" the same)
- $(A \cup B) \times C = A \times C \cup B \times C$. Also, $(A \cap B) \times C = A \times C \cap B \times C$
  - $(A \cup B) \times (C \cup D) = A \times (C \cup D) \cup B \times (C \cup D) = A \times C \cup A \times D \cup B \times C \cup B \times D$
- Complement: $\overline{S \times T} = ?$
  - $\overline{S} \times \overline{T} \cup \overline{S} \times T \cup S \times \overline{T}$

USRT

Question

- Let $S, T \subseteq \mathbb{Z}$. Pick the best choice

A. $S \subseteq S \times T$

B. $S \cap T \subseteq S \times T$

C. $S \cup T \subseteq S \times T$

D. $S \subseteq S \times T \quad \leftrightarrow \quad S = \emptyset$

E. None of the above

Relations

Relations: Basics

- A relation between elements in a set S is technically a subset of S×S, namely the pairs for which the relation holds
  - Or a predicate over the domain S×S
  - e.g. Likes(x,y)
  - Likes = { (Alice,Alice),
    (Alice, Flamingo),
    (J'wock,J'wock),
    (Flamingo,Flamingo) }
  - More common notation:
    x Likes y
    - or, x⊏y, x ≥ y, x~y, xLy, ...

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

Sets & Relations in action

Relational Database

x	y	Likes(x,y)
Alice	Alice	TRUE
	Jabberwock	FALSE
	Flamingo	TRUE
Jabberwock	Alice	FALSE
	Jabberwock	TRUE
	Flamingo	FALSE
Flamingo	Alice	FALSE
	Jabberwock	FALSE
	Flamingo	TRUE

Relational DB Table

Likes	
x	y
Alice	Alice
Alice	Flamingo
Jabberwock	Jabberwock
Flamingo	Flamingo

- Queries to the DB are set/logical operations
  - SELECT x
    FROM Likes
    WHERE y='Alice' OR y='Flamingo'
  - $\{ x \mid (x,Alice) \in Likes \} \cup \{ x \mid (x,Flamingo) \in Likes \}$

What is a Relation?

Many ways to look at it!

$R \subseteq S \times S$

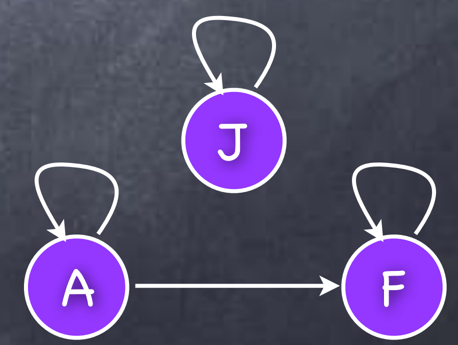
a set of ordered-pairs $\{ (a,b) \mid a \sqsubset b \}$

{ (A,A), (A,F), (J,J), (F,F) }

Boolean matrix, $M_{a,b} = 1$ iff $a \sqsubset b$

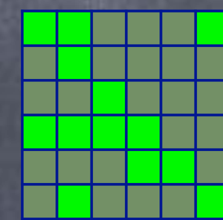
	A	J	F
A	1	0	1
J	0	1	0
F	0	0	1

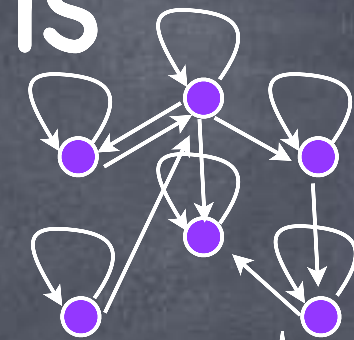
(directed) graph



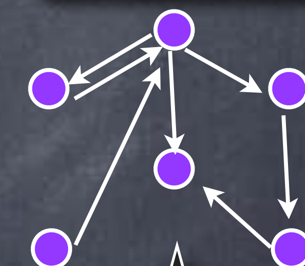
(Ir)Reflexive Relations

- Reflexive (e.g. Knows, ≤)
  - The kind of relationship that everyone has with themselves





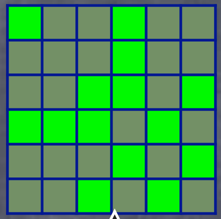
- Irreflexive (e.g. Gave birth to, ≠)
  - The kind that nobody has with themselves



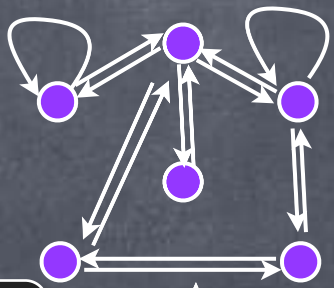
- Neither (e.g. is a prime factor of)
  - Some, but not all, have this relationship with themselves

(Anti)Symmetric Relations

- Symmetric (e.g. sits next to)
  - The relationship is reciprocated



symmetric matrix



self-loops & bidirectional edges only

- Anti-symmetric (e.g. Parent of, divides (in $\mathbb{Z}^+$), $<$)
  - No reciprocation (except possibly with self)

no bidirectional edges

- Neither (e.g. in the "circle" of)
  - Reciprocated in some pairs (with distinct members) and only one-way in other pairs

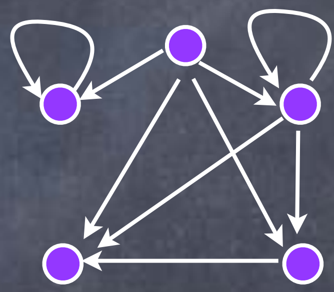
some bidirectional, some unidirectional

- Both (e.g., =)
  - Each one related only to self (if at all)

no edges except self-loops

Transitive Relations

- Transitive (e.g., Ancestor of, subset of, divides, ≤)
  - if a is related to b and b is related to c, then a is related to c



if there is a "path" from a to z, then there is edge (a,z)

  - "Transitive closure" of the relation is same as itself
- Intransitive: Not transitive