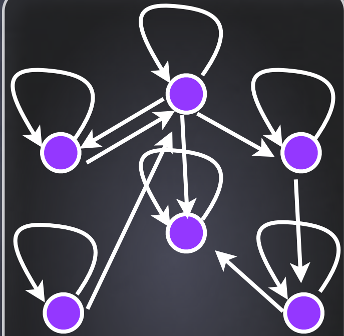
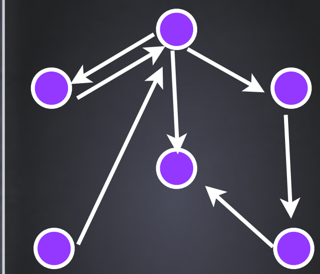
Relations

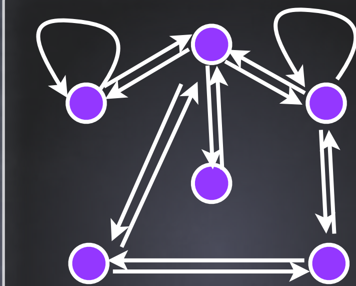
Lecture 9



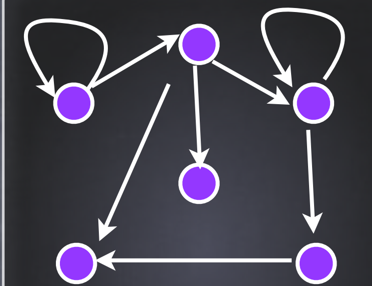
Reflexive:
All self-loops



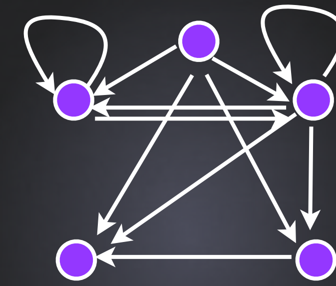
Irreflexive:
No self-loops



Symmetric:
Only self-loops & bidirectional edges



Anti-symmetric:
No bidirectional edges



Transitive:
Path from a to b implies edge (a,b)

The complete relation R = S × S is reflexive, symmetric and transitive

Reflexive closure of R: Smallest relation R′ ⊇ R s.t. R′ is reflexive
Symmetric closure of R: Smallest relation R′ ⊇ R s.t. R′ is symmetric
Transitive closure of R: Smallest relation R′ ⊇ R s.t. R′ is transitive
Each of these is unique

Question

02:00

- Let $\sqsubset$ be the empty relation (i.e., $\forall a,b \; \neg(a \sqsubset b)$). Choose the best option.

  A. $\sqsubset$ is transitive

  B. $\sqsubset$ is irreflexive

  C. $\sqsubset$ is symmetric

  D. All of the above

  E. Some of the above (but not all)

All. Also, anti-symmetric.

Question

- Let $\sqsubset$ be the relation over integers defined as $x \sqsubset y$ if $|x-y| \leq 10$. Choose the best option.

A. $\sqsubset$ is transitive

B. $\sqsubset$ is reflexive

C. $\sqsubset$ is symmetric

D. All of the above

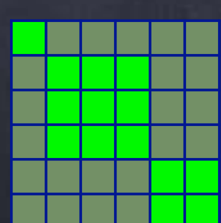
E. Some of the above (but not all)

Not transitive

Equivalence Relation

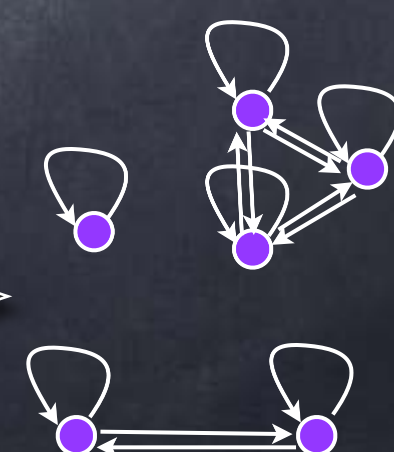
- A relation that is reflexive, symmetric and transitive
  - e.g. is a relative, has the same last digit, is congruent mod 7, ...
- Claim: Let $Eq(x) \triangleq \{y|x \sim y\}$. If $Eq(x) \cap Eq(y) \neq \emptyset$, then $Eq(x) = Eq(y)$.
  - Let $z \in Eq(x) \cap Eq(y)$. $\forall w \in Eq(x)$, $x \sim w$. Also, $x \sim z \Rightarrow w \sim z$. Also, $y \sim z \Rightarrow y \sim w \Rightarrow w \in Eq(y)$. i.e., $Eq(x) \subseteq Eq(y)$.
- The Equivalence classes partition the domain

$P_1,..,P_t \subseteq S$
s.t.
$P_1 \cup .. \cup P_t = S$
$P_i \cap P_j = \emptyset$



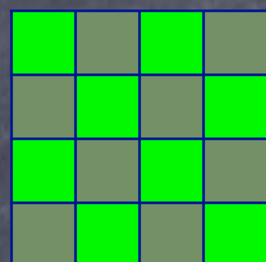
Square blocks along the diagonal, after sorting the elements by equivalence class

"Cliques" for each class

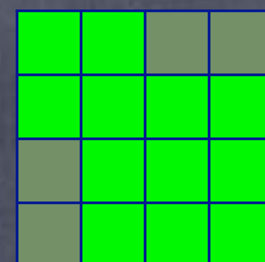


Question

- Which one(s) represent(s) equivalence relation(s)

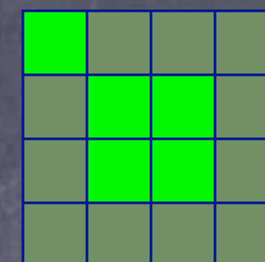


R_1



R_2

not transitive



R_3

not reflexive

A. R_1 and R_3
B. R_1 only
C. R_2 only
D. R_3 only
E. None of the above

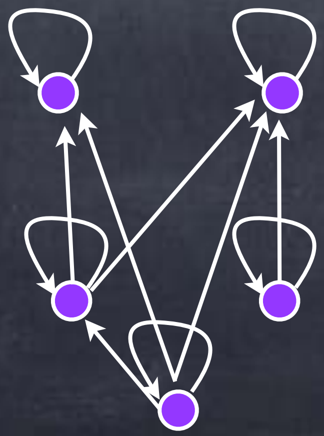
Posets

Strict partial order: irreflexive, rather than reflexive

- Partial order: a transitive, anti-symmetric and reflexive relation
  - e.g. $\leq$ for integers, divides for integers, $\subseteq$ for sets, "containment" for line-segments
  - Partial: Some pair may be "incomparable"
- Transitive and anti-symmetric $\rightarrow$ "acyclic"
- Partially ordered set (a.k.a Poset): a set and a partial order over it

Cyclic: Some node s.t. you can leave it through an edge (not self-loop), move through some edges, and return to the node

$S_1=\{0,1,2,3\}$, $S_2=\{1,2,3,4\}$, $S_3=\{1,2\}$, $S_4=\{3,4\}$, $S_5 = \{2\}$.
Relation $\subseteq$



Check:
- Anti-symmetric (no bidirectional edges),
- Transitive,
- Reflexive (all self-loops)

Posets

Do exist in finite posets (Prove by induction on $|S|$)

- Maximal & minimal elements of a poset $(S, \preccurlyeq)$
  - $x \in S$ is maximal if $\nexists y \in S-\{x\}$ s.t. $x \preccurlyeq y$
  - $x \in S$ is minimal if $\nexists y \in S-\{x\}$ s.t. $y \preccurlyeq x$
  - Need not exist (e.g., in $(\mathbb{Z}, \leq)$). Need not be unique when it exists (e.g., $(S, \subseteq)$, where S is the set of all subsets of integers that have at least one odd number)
- Greatest element in $T \subseteq S$: $x \in T$ s.t. $\forall y \in T$, $y \preccurlyeq x$

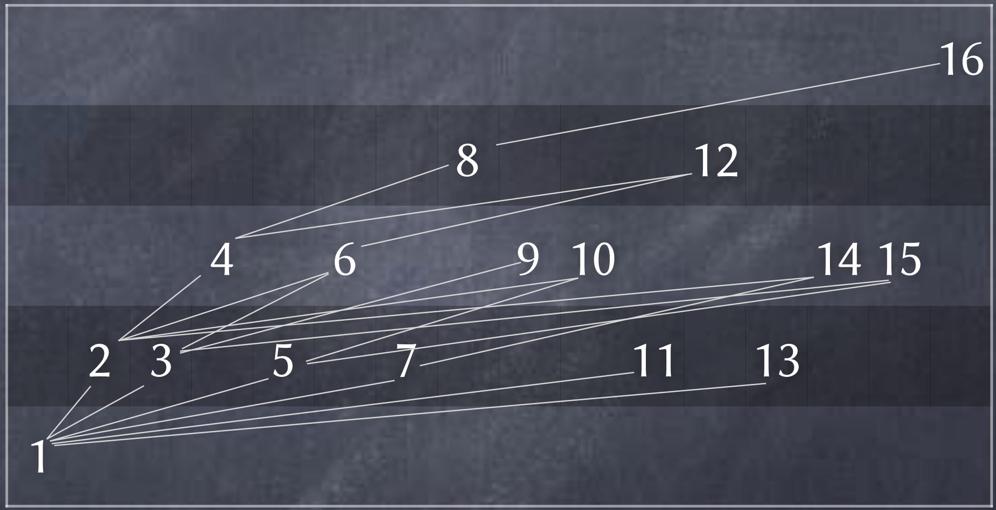
  Least element in $T \subseteq S$: $x \in T$ s.t. $\forall y \in T$, $x \preccurlyeq y$
  - Need not exist, even if T finite. Unique when it exists.
- Upper Bound for $T \subseteq S$: x s.t. $\underline{\forall y \in T, y \preccurlyeq x}$

  Lower Bound for $T \subseteq S$: x s.t. $\underline{\forall y \in T, x \preccurlyeq y}$
- Least Upper Bound for T: Least in $\{x \mid x \text{ u.b. for } T\}$

  Greatest Lower Bound for T: Greatest in $\{x \mid x \text{ l.b. for } T\}$

An Example

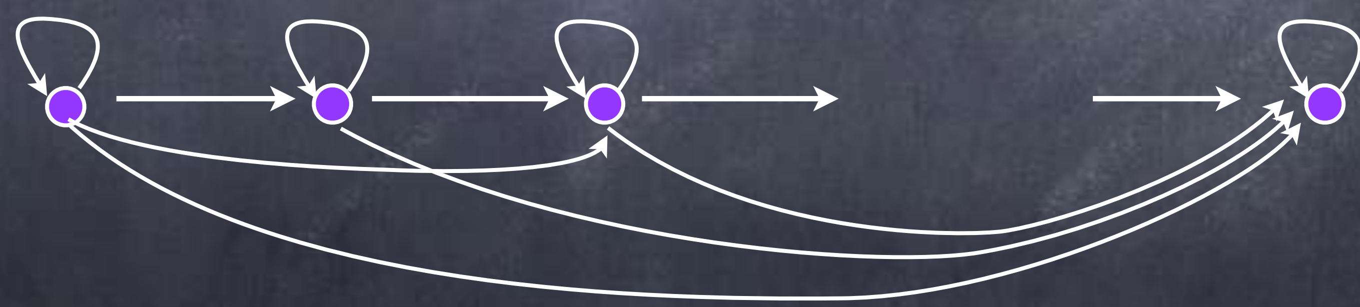
- Let $a \sqsubset b$ iff b/a is prime (with $\mathbb{Z}^+$ as the domain)
- Let $\preccurlyeq$ be the transitive and reflexive closure of $\sqsubset$
  - $a \preccurlyeq b$ iff $a|b$
- Divisibility poset: $(\mathbb{Z}^+, \preccurlyeq)$
- When is c a lower bound for $T=\{a,b\}$? $c \preccurlyeq a$ and $c \preccurlyeq b$.



  - c is a common divisor for $\{a,b\}$.
  - $\gcd(a,b)$ = greatest lower bound for $\{a,b\}$ in this posey

Total/Linear Order

- In some posets every two elements are “comparable”: for {a,b}, either $a \sqsubseteq b$ or $b \sqsubseteq a$
    - Can arrange all the elements in a line, with <u>all possible</u> right-pointing edges (plus, self-loops)



- If finite, has <u>unique</u> maximal and <u>unique</u> minimal elements (left and right ends)

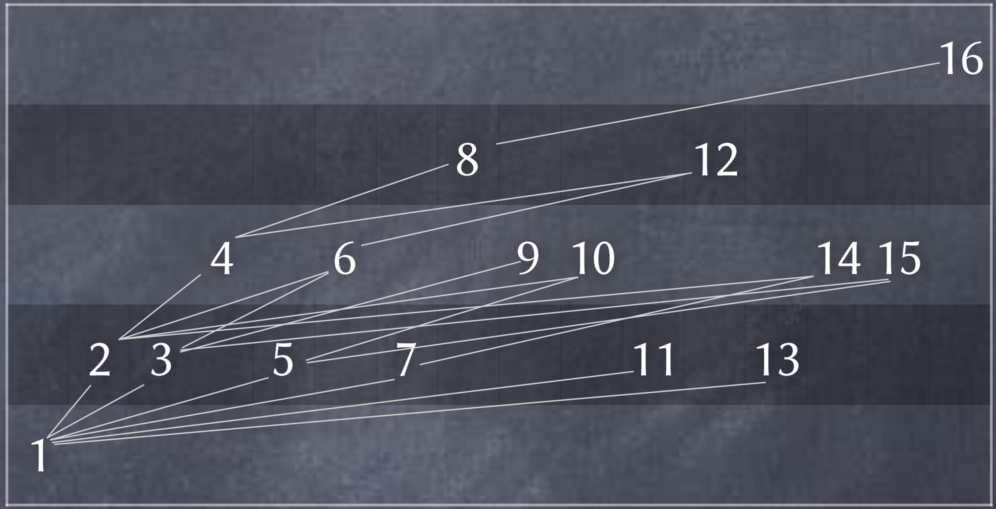
Order Extension

- A poset P'=(S,$\leq$) is an extension of a poset P=(S,$\preccurlyeq$) if $\forall a,b \in S$, $a \preccurlyeq b \rightarrow a \leq b$
- Any finite poset can be extended to a total ordering (this is called topological sorting)
  - By induction on |S|
    - Induction step: Remove a minimal element, extend to a total ordering, reintroduce the removed element as the minimum in the total ordering.
  - For infinite posets? The "Order Extension Principle" is typically taken as an axiom! (Unless an even stronger axiom called the "Axiom of Choice" is used)

Finite if S is finite

Chains

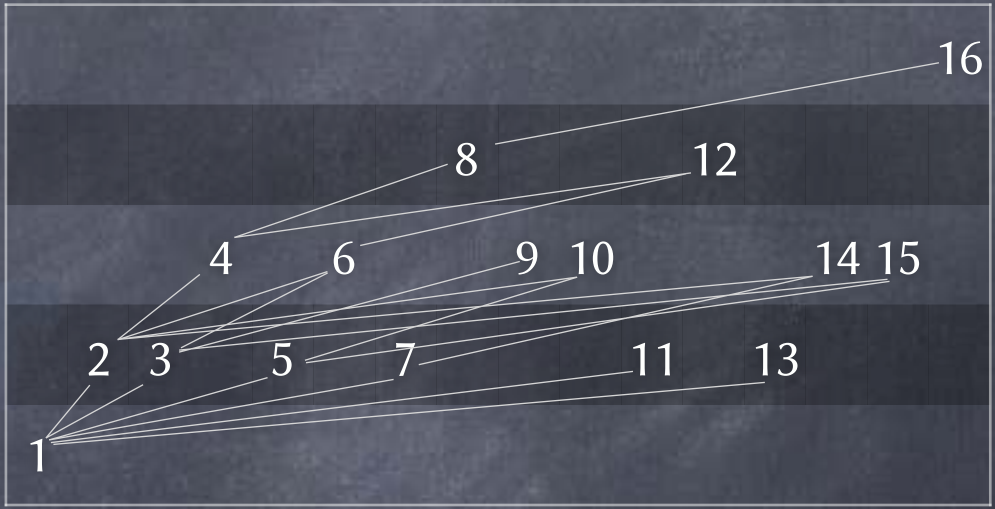
- $C \subseteq S$ is called a chain if $\forall$ a,b $\in$ C, either $a \preccurlyeq b$ or $b \preccurlyeq a$
- That is, $(C, \preccurlyeq)$ is a total order
- Every element $a \in S$ belongs to some chain in which it is the maximum element (possibly just {a})



- Height(a) = max length chain with a as the maximum
- E.g., In "Divisibility poset," height(1)=1, height(p)=2 for all primes p. For $m = p_1^{d_1} \cdot \ldots \cdot p_t^{d_t}$ (p_i primes) height(m) = $1 + \Sigma_i d_i$

Anti-Chains

- $A \subseteq S$ is called an anti-chain if $\forall a,b \in A, \quad a \neq b \longrightarrow$ neither $a \preccurlyeq b$ nor $b \preccurlyeq a$
- $(A, \preccurlyeq)$ is the equality relation
- Let $A_h = \{ a \mid \text{height}(a)=h \}$
- For every finite h, A_h is an anti-chain (possibly empty)



  - Otherwise, $\exists a \neq b$, $a \preccurlyeq b$ with $\text{height}(a) = \text{height}(b) = h$.
    $\text{height}(a) = h \Rightarrow \exists$chain C s.t. $a=\max(C)$ and $|C|=h$

How?

$\Rightarrow b \notin C$ and $C' = C \cup \{b\}$ is a chain with $b = \max(C')$

$\Rightarrow \text{height}(b) \geq h+1$!

Anti-Chains

- $A \subseteq S$ is called an anti-chain if $\forall a,b \in A, \quad a \neq b \rightarrow$ neither $a \preccurlyeq b$ nor $b \preccurlyeq a$
- $(A,\preccurlyeq)$ is the equality relation
- Let $A_h = \{ a \mid \text{height}(a)=h \}$
- For every finite h, A_h is an anti-chain (possibly empty)

- In a finite poset, since every element has a finite height, every element appears in some A_h: i.e., A_hs partition S
- Mirsky's Theorem: Least number of anti-chains needed to partition S is exactly the length of a longest chain

Height of the poset