Relations

Lecture 9
The complete relation $R = S \times S$ is reflexive, symmetric and transitive.

**Reflexive closure** of $R$: Smallest relation $R' \supseteq R$ s.t. $R'$ is reflexive

**Symmetric closure** of $R$: Smallest relation $R' \supseteq R$ s.t. $R'$ is symmetric

**Transitive closure** of $R$: Smallest relation $R' \supseteq R$ s.t. $R'$ is transitive

Each of these is unique.
Let ⊏ be the empty relation (i.e., ∀a,b ¬(a ⊏ b)). Choose the best option.

A. ⊏ is transitive
B. ⊏ is irreflexive
C. ⊏ is symmetric
D. All of the above
E. Some of the above (but not all)

All. Also, anti-symmetric.
Let $\sqsubseteq$ be the relation over integers defined as $x \sqsubseteq y$ if $|x-y| \leq 10$. Choose the best option.

A. $\sqsubseteq$ is transitive
B. $\sqsubseteq$ is reflexive
C. $\sqsubseteq$ is symmetric
D. All of the above
E. Some of the above (but not all)
Equivalence Relation

- A relation that is reflexive, symmetric and transitive
- e.g. is a relative, has the same last digit, is congruent mod 7, ...

Claim: Let \( Eq(x) \triangleq \{ y \mid x \sim y \} \). If \( Eq(x) \cap Eq(y) \neq \emptyset \), then \( Eq(x) = Eq(y) \).

Let \( z \in Eq(x) \cap Eq(y) \). \( \forall w \in Eq(x) \), \( x \sim w \). Also, \( x \sim z \Rightarrow w \sim z \).
Also, \( y \sim z \Rightarrow y \sim w \Rightarrow w \in Eq(y) \). i.e., \( Eq(x) \subseteq Eq(y) \).

The Equivalence classes partition the domain.

\[ P_1, \ldots, P_t \subseteq S \]
\[ s.t. \]
\[ P_1 \cup \ldots \cup P_t = S \]
\[ P_i \cap P_j = \emptyset \]

Square blocks along the diagonal, after sorting the elements by equivalence class.

“Cliques” for each class.
Question

Which one(s) represent(s) equivalence relation(s)

A. $R_1$ and $R_3$
B. $R_1$ only
C. $R_2$ only
D. $R_3$ only
E. None of the above
**Posets**

- **Partial order**: a transitive, anti-symmetric and reflexive relation
  - e.g. $\leq$ for integers, divides for integers, $\subseteq$ for sets, "containment" for line-segments
  - **Partial**: Some pair may be "incomparable"

- Transitive and anti-symmetric $\rightarrow$ "acyclic"

- Partially ordered set (a.k.a Poset): a set and a partial order over it

  - Check:
    - Anti-symmetric (no bidirectional edges),
    - Transitive,
    - Reflexive (all self-loops)

**Examples**

- $S_1=\{0,1,2,3\}$, $S_2=\{1,2,3,4\}$, $S_3=\{1,2\}$, $S_4=\{3,4\}$, $S_5=\{2\}$,
- Relation $\subseteq$

**Strict partial order**: irreflexive, rather than reflexive

**Cyclic**: Some node s.t. you can leave it through an edge (not self-loop), move through some edges, and return to the node
Posets

- Maximal & minimal elements of a poset \((S, \preceq)\)
  - \(x \in S\) is maximal if \(\nexists y \in S - \{x\}\) s.t. \(x \preceq y\)
  - \(x \in S\) is minimal if \(\nexists y \in S - \{x\}\) s.t. \(y \preceq x\)
  - Need not exist (e.g., in \((\mathbb{Z}, \leq)\)). Need not be unique when it exists (e.g., \((S, \subseteq)\), where \(S\) is the set of all subsets of integers that have at least one odd number)

- Greatest element in \(T \subseteq S\): \(x \in T\) s.t. \(\forall y \in T, y \preceq x\)
- Least element in \(T \subseteq S\): \(x \in T\) s.t. \(\forall y \in T, x \preceq y\)
  - Need not exist, even if \(T\) finite. Unique when it exists.

- Upper Bound for \(T \subseteq S\): \(x\) s.t. \(\forall y \in T, y \preceq x\)
- Lower Bound for \(T \subseteq S\): \(x\) s.t. \(\forall y \in T, x \preceq y\)

- Least Upper Bound for \(T\): Least in \(\{x\mid x \text{ u.b. for } T\}\)
- Greatest Lower Bound for \(T\): Greatest in \(\{x\mid x \text{ l.b. for } T\}\)

Do exist in finite posets (Prove by induction on \(|S|\))
An Example

- Let $a \sqsubseteq b$ iff $b/a$ is prime (with $\mathbb{Z}^+$ as the domain)

- Let $\leq$ be the transitive and reflexive closure of $\sqsubseteq$
  - $a \leq b$ iff $a|b$

- Divisibility poset: $(\mathbb{Z}^+, \leq)$

- When is $c$ a lower bound for $T=\{a,b\}$? $c \leq a$ and $c \leq b$.

- $c$ is a common divisor for $\{a,b\}$.

- $\gcd(a,b) =$ greatest lower bound for $\{a,b\}$ in this posey
In some posets every two elements are “comparable”: for \{a, b\}, either \( a \leq b \) or \( b \leq a \).

Can arrange all the elements in a line, with all possible right-pointing edges (plus, self-loops).

If finite, has unique maximal and unique minimal elements (left and right ends).
Order Extension

A poset $P'=(S,\leq)$ is an extension of a poset $P=(S,\preceq)$ if
\[ \forall a, b \in S, \ a \preceq b \rightarrow a \leq b \]

Any finite poset can be extended to a total ordering (this is called topological sorting)

By induction on $|S|$

Induction step: Remove a minimal element, extend to a total ordering, reintroduce the removed element as the minimum in the total ordering.

For infinite posets? The “Order Extension Principle” is typically taken as an axiom! (Unless an even stronger axiom called the “Axiom of Choice” is used)
Chains

- $C \subseteq S$ is called a chain if $\forall a,b \in C$, either $a \leq b$ or $b \leq a$

- That is, $(C, \leq)$ is a total order

- Every element $a \in S$ belongs to some chain in which it is the maximum element (possibly just $\{a\}$)

- Height$(a) = \max$ length chain with $a$ as the maximum

- E.g., In “Divisibility poset,” height$(1)=1$, height$(p)=2$ for all primes $p$. For $m=p_1^{d_1} \cdots p_t^{d_t}$ ($p_i$ primes) $\text{height}(m) = 1+\sum_i d_i$

Finite if $S$ is finite
Anti-Chains

A ⊆ S is called an anti-chain if ∀a,b ∈ A, a ≠ b → neither a ≼ b nor b ≼ a

(A, ≼) is the equality relation

Let A_h = { a | height(a) = h }

For every finite h, A_h is an anti-chain (possibly empty)

Otherwise, ∃ a ≠ b, a ≼ b with height(a) = height(b) = h.

height(a) = h ⇒ ∃ chain C s.t. a = max(C) and |C| = h

How? ⇒ b ∉ C and C' = C ∪ {b} is a chain with b = max(C')
⇒ height(b) ≥ h + 1!
Anti-Chains

- $A \subseteq S$ is called an anti-chain if
  $\forall a, b \in A, \quad a \not= b \rightarrow \text{neither } a \preceq b \text{ nor } b \preceq a$

- $(A, \preceq)$ is the equality relation

- Let $A_h = \{ a \mid \text{height}(a) = h \}$

- For every finite $h$, $A_h$ is an anti-chain (possibly empty)

- In a finite poset, since every element has a finite height, every element appears in some $A_h$: i.e., $A_h$s partition $S$

- Mirsky's Theorem: Least number of anti-chains needed to partition $S$ is exactly the length of a longest chain