Functions

Lecture 10

Functions

- For each element in a universe (domain), a predicate assigns one of two values, True and False.
- "Co-domain" is {True,False}
- Functions: more general co-domains
  - $f : A \longrightarrow B$
- A function maps each element in the domain to an element in the co-domain
- To specify a function, should specify domain, co-domain and the "table" itself

pair∈AIW^2	Likes(pair)
(Alice, Alice)	TRUE
(Alice, Jabberwock)	FALSE
(Alice, Flamingo)	TRUE
(Jabberwock, Alice)	FALSE
(Jabberwock, Jabberwock)	TRUE
(Jabberwock, Flamingo)	FALSE
(Flamingo, Alice)	FALSE
(Flamingo, Jabberwock)	FALSE
(Flamingo, Flamingo)	TRUE

Function

- A function maps each element in the domain to an element in the co-domain
  - eg: Extent of liking, $f: AIW^2 \longrightarrow \{0,1,2,3,4,5\}$
  - Note: no empty slot, no slot with more than one entry
  - Not all values from the co-domain need be used
- Image: set of values in the co-domain that do get used
  - For $f: A \longrightarrow B$, $Im(f) \subseteq B$ s.t. $Im(f) = \{ y \in B \mid \exists x \in A \;\; f(x) = y \}$

x∈Domain	f(x)∈Co-Domain
(Alice, Alice)	5
(Alice, Jabberwock)	1
(Alice, Flamingo)	4
(Jabberwock, Alice)	0
(Jabberwock, Jabberwock)	4
(Jabberwock, Flamingo)	0
(Flamingo, Alice)	1
(Flamingo, Jabberwock)	0
(Flamingo, Flamingo)	5

Function

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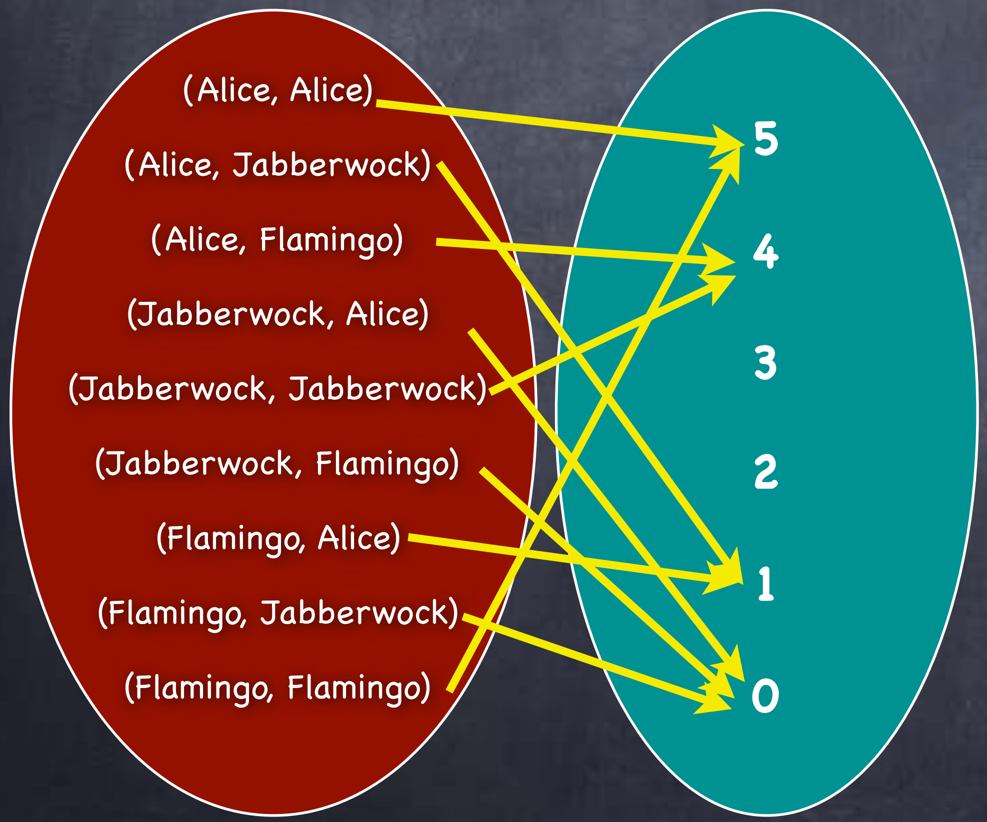
  - eg: Extent of liking, f: $AIW^2 \longrightarrow \{0,1,2,3,4,5\}$



x∈Domain	f(x)∈Co-Domain
(Alice, Alice)	5
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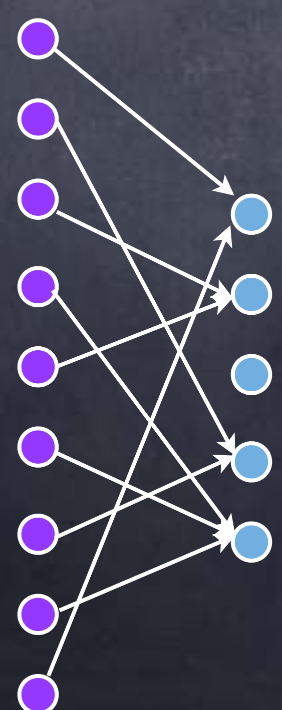
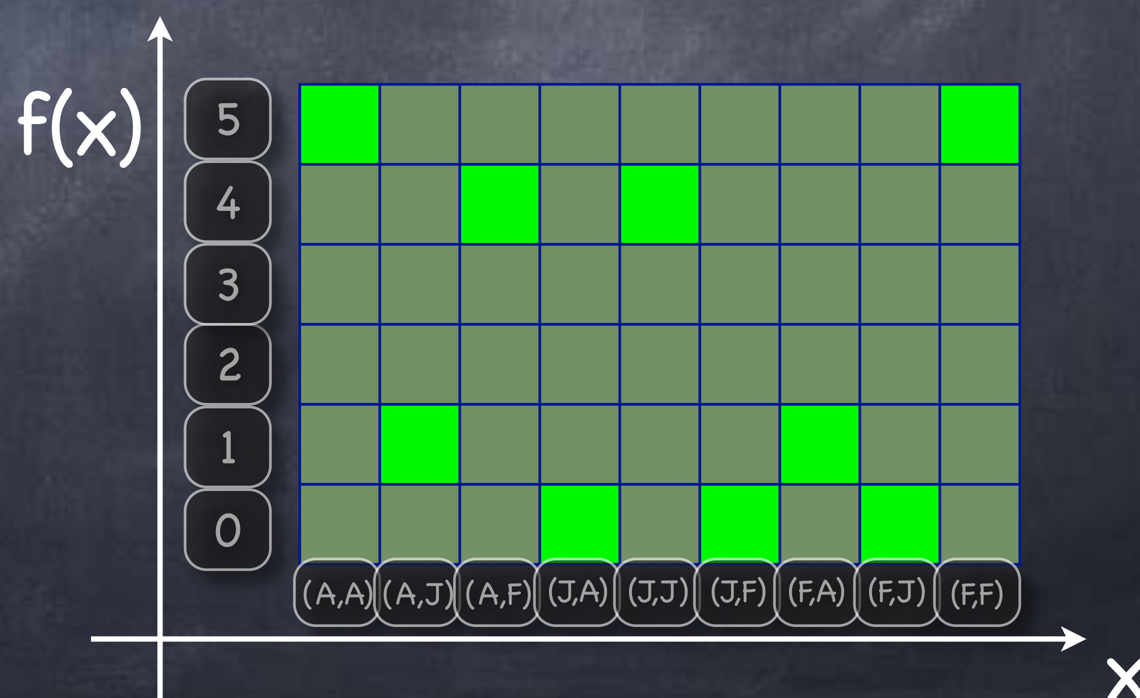
Function as a Relation

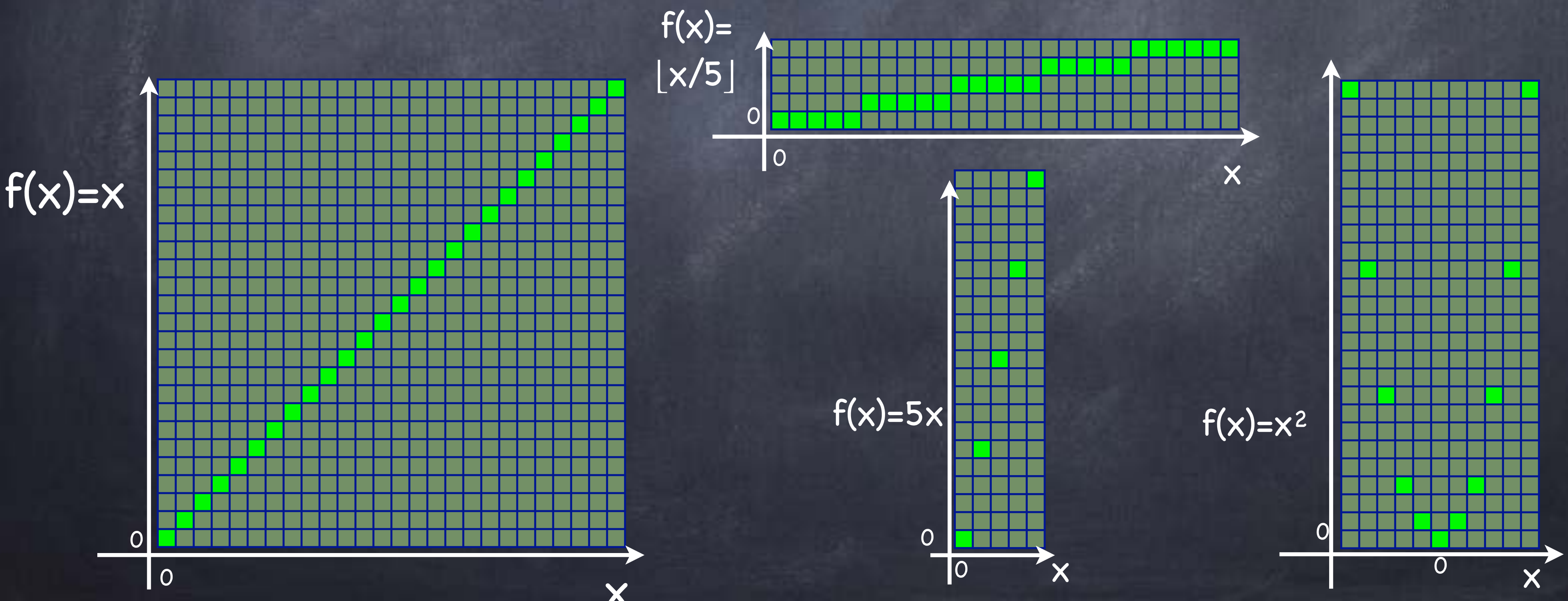
- As a relation between domain & co-domain, $R_f \in$ domain $\times$ co-domain
  $R_f = \{ (x,f(x)) \mid x \in \text{domain} \}$
  - Special property of R_f: every x has a unique y s.t. $(x,y) \in R_f$
- Can be represented using a matrix
  - Convention: domain on the "x-axis", co-domain on the "y-axis"
  - Every column has exactly one cell "switched on"





Plotting a Function

- When both domain and co-domain are numerical (or otherwise totally ordered), we often "plot" the function
  - Shows only part of domain/codomain when they are infinite (here f:$\mathbb{Z} \to \mathbb{Z}$)

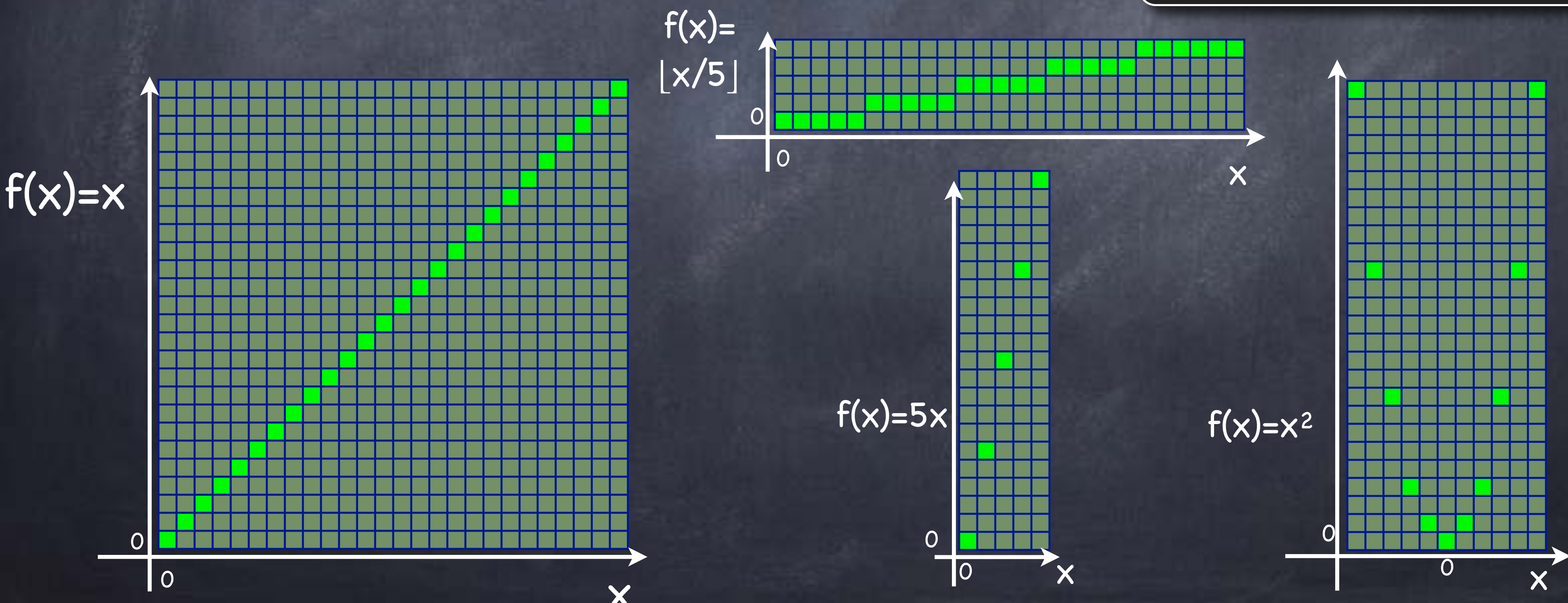


Types of Functions

- Function: every column has exactly one cell “on”
- Onto Function (surjection): Every row has <u>at least</u> one cell “on”
- One-to-One function (injection): Every row has <u>at most</u> one cell “on”
- Bijection: Every row has <u>exactly</u> one cell “on”

$\forall y\ \exists x\ f(x)=y$

$\forall y \in Im(f)\ \exists!\ x \in A\quad f(x)=y$



Question

02:00

- Let $f: \mathbb{N} \to \mathbb{N}$, $g: \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $h: \mathbb{N} \to \mathbb{R}_{\geq 0}$ be defined as: $f(x) = x^2$, $g(x) = x^2$, $h(x) = x^2$. Which ones are onto?

A. f, g and h

B. f and g

C. only f

D. only g

E. None of the above

h not onto, since (say) $2 \in \text{Codomain}(h) - \text{Im}(h)$

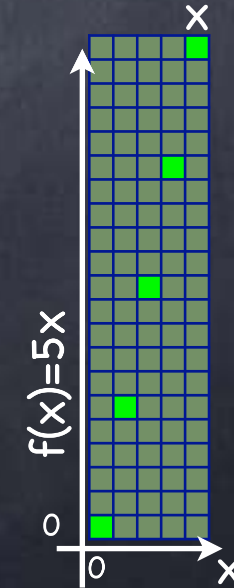
g is onto (every non-negative real number has a square-root)

f not onto, since (say) $2 \in \text{Codomain}(f) - \text{Im}(f)$

Injective Functions

- A function f:A→B is one-to-one if $\forall x,x' \in A \quad f(x)=f(x') \rightarrow x=x'$
- $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x)=x^2$ is not one-to-one
  - $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined as $f(x)=x^2$ is one-to-one
- $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x)=\lfloor x/5 \rfloor$ is not one-to-one

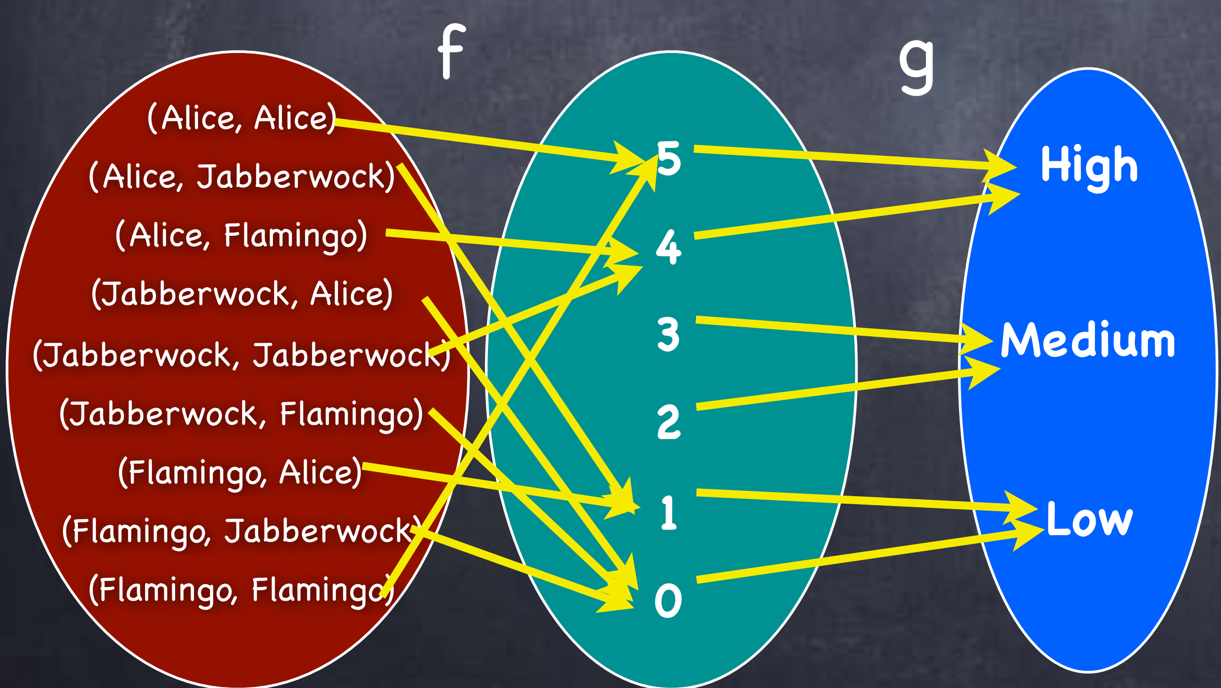
  - $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x)=5x$ is one-to-one

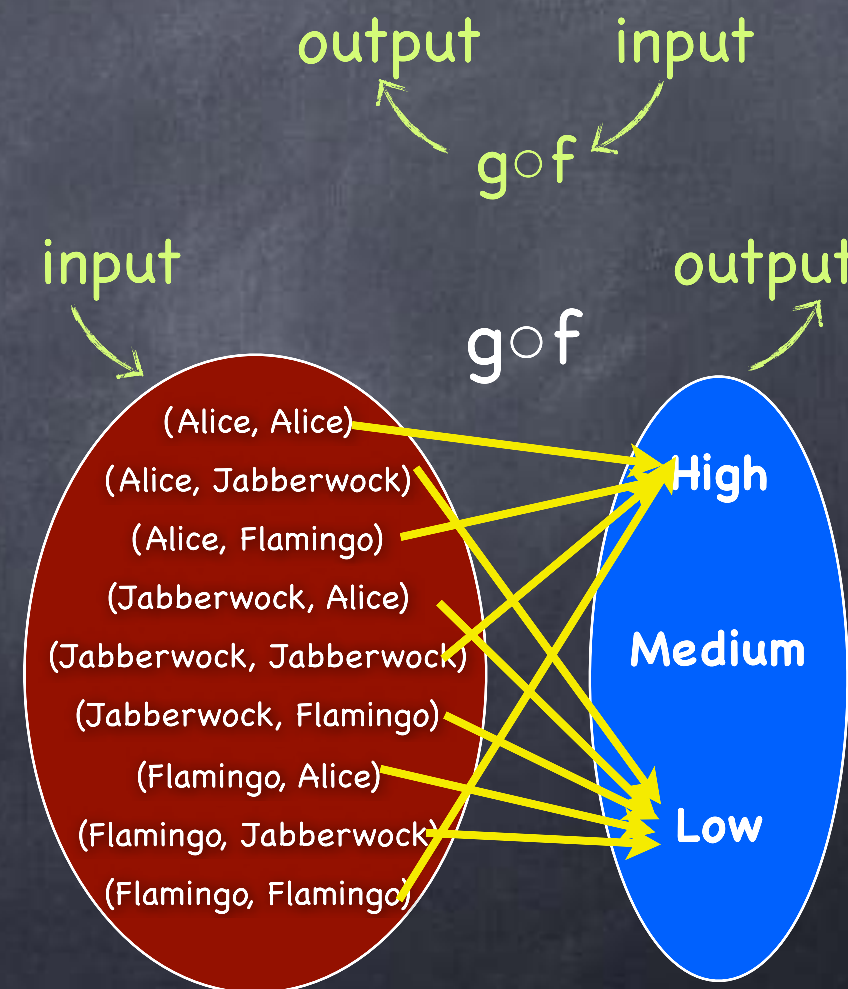


- In fact, any strictly increasing function is one-to-one
  - And, any strictly decreasing function too is one-to-one
- One-to-one functions don't lose any information
  - They are "invertible"

Composition

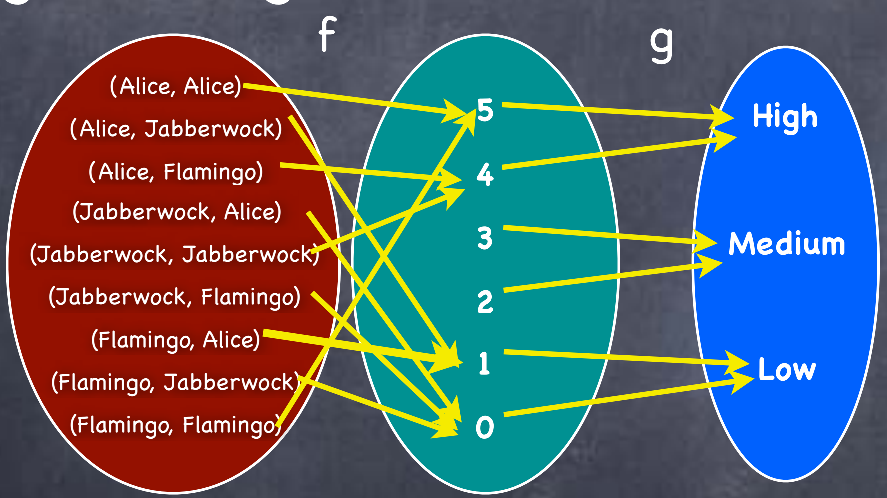
- Composition of functions f and g: $g \circ f$: Domain(f) $\longrightarrow$ Co-domain(g)
  - $g \circ f(x) \triangleq g(f(x))$

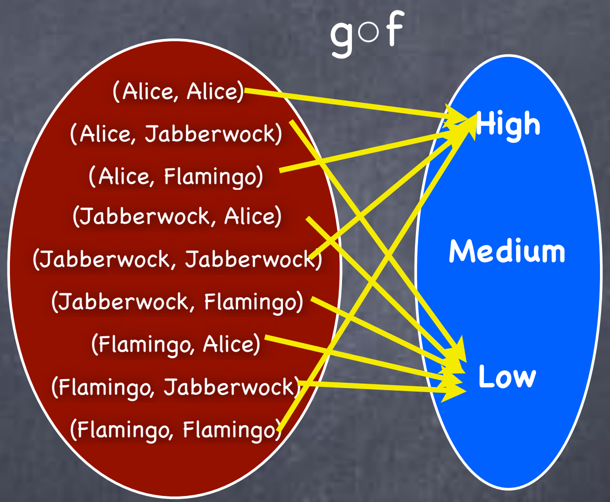




Composition

- Composition of functions f and g: g∘f : Domain(f) ⟶ Co-domain(g)
  - g∘f(x) ≜ g(f(x))



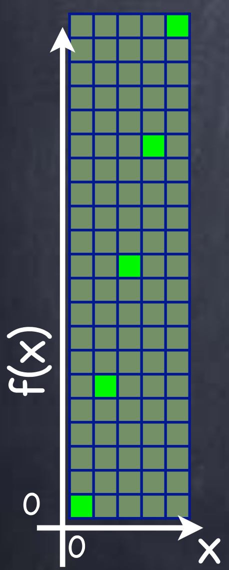


- Defined only if Im(f) ⊆ Domain(g)
  - Typically, Domain(g) = Co-domain(f)
- g∘f : Domain(f) ⟶ Co-domain(g)
- Im(g∘f) ⊆ Im(g)

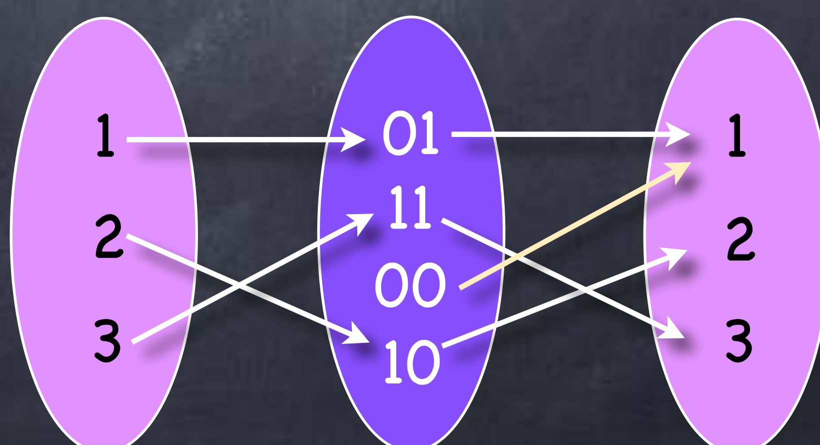
Injective ⟷ Invertible

- f is said to be invertible if $\exists g$ s.t. $g \circ f \equiv Id$
- One-to-one functions are invertible
- Suppose $f : A \longrightarrow B$ is one-to-one

$\forall y \in Im(f)\ \exists! x \in A\quad f(x) = y$



  - Let $g : B \longrightarrow A$ be defined as follows:
    for $y \in Im(f)$, $g(y)=x$ s.t. $f(x)=y$ (well-defined)
    for $y \notin Im(f)$, $g(y)$ = some arbitrary element in A
    - Then $g \circ f \equiv Id_A$, where $Id_A : A \longrightarrow A$ is the identity function over A
    - g need not be invertible



Injective $\longleftrightarrow$ Invertible

- f is said to be invertible if $\exists g$ s.t. $g \circ f \equiv Id$
- One-to-one functions are invertible
- And invertible functions are one-to-one
  - Suppose $f : A \rightarrow B$ is invertible
  - Let $g : B \rightarrow A$ be s.t. $g \circ f \equiv Id$
  - Now, for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$
  - But $g(f(x)) = Id(x) = x$
  - Hence, $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

Question

- Suppose A, B are finite sets such that $|A| < |B|$.

  Suppose $f : A \longrightarrow B$. Then:

A. f can be onto, but not one-to-one

B. f can be one-to-one, but not onto

Onto $\longrightarrow |B| \leq |A|$

C. f can neither be onto nor one-to-one

D. f may be either onto or one-to-one, or both

E. f may be either onto or one-to-one, but not both

Question

02:00

- Suppose A, B are finite sets such that |A| > |B|.

  Suppose f : A ⟶ B. Then:

  A. f can be onto, but not one-to-one

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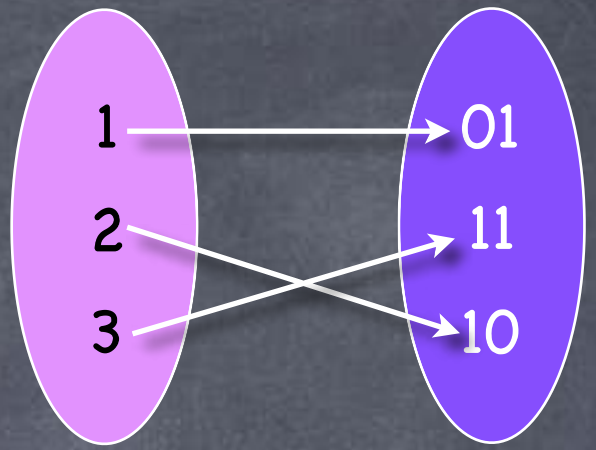
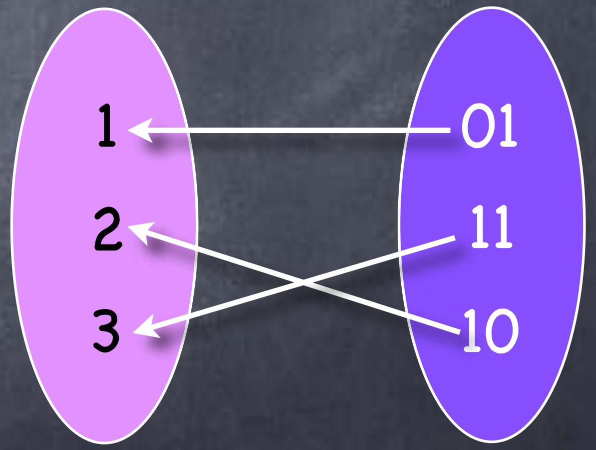
<u>Pigeonhole Principle</u>
|A| > |B| ⟶
f not one-to-one.

One-to-one ⟶ |A|≤|B|

Bijections



- Bijection: both onto and one-to-one
  - Every row and every column has exactly one cell "on"
  - Every element in the co-domain has exactly one "pre-image"
    - If $f : A \rightarrow B$, $f^{-1} : B \rightarrow A$ such that $f^{-1} \circ f : A \rightarrow A$ and $f \circ f^{-1} : B \rightarrow B$ are both identity functions
    - Both f and f^{-1} are invertible, and the inverses are unique
    - $(f^{-1})^{-1} = f$



- If A, B finite sets and there is a bijection $f:A \rightarrow B$, then $|A|=|B|$
- If A, B finite sets and $|A|=|B|$ and $f:A \rightarrow B$, then f is onto $\equiv$ f in one-to-one $\equiv$ f is a bijection

Composition & Onto/One-to-One

- Composition "**respects onto-ness**"
  - If f and g are onto, $g \circ f$ is onto as well
  - If $g \circ f$ is onto, then g is onto
- Composition "**respects one-to-one-ness**"
  - If f and g are one-to-one, $g \circ f$ is one-to-one as well
  - If $g \circ f$ is one-to-one, then f is one-to-one
- Hence, composition **respects bijections**
  - If f and g are bijections then $g \circ f$ is a bijection as well
  - If $g \circ f$ is a bijection, then f is one-to-one and g is onto

With the convention Domain(g)=Co-Domain(f)

Permutation of a string

- To permute = to rearrange
  - e.g., π_{53214}(hello) = lleoh
  - e.g., π_{35142}(lleoh) = ehlol
- Permutations are essentially bijections from the set of positions (here {1,2,3,4,5}) to itself
  - A bijection from any finite set to itself is called a permutation
- Permutations compose to yield permutations (since bijections do so)
  - e.g., $\pi_{35142} \circ \pi_{53214} = \pi_{21534}$