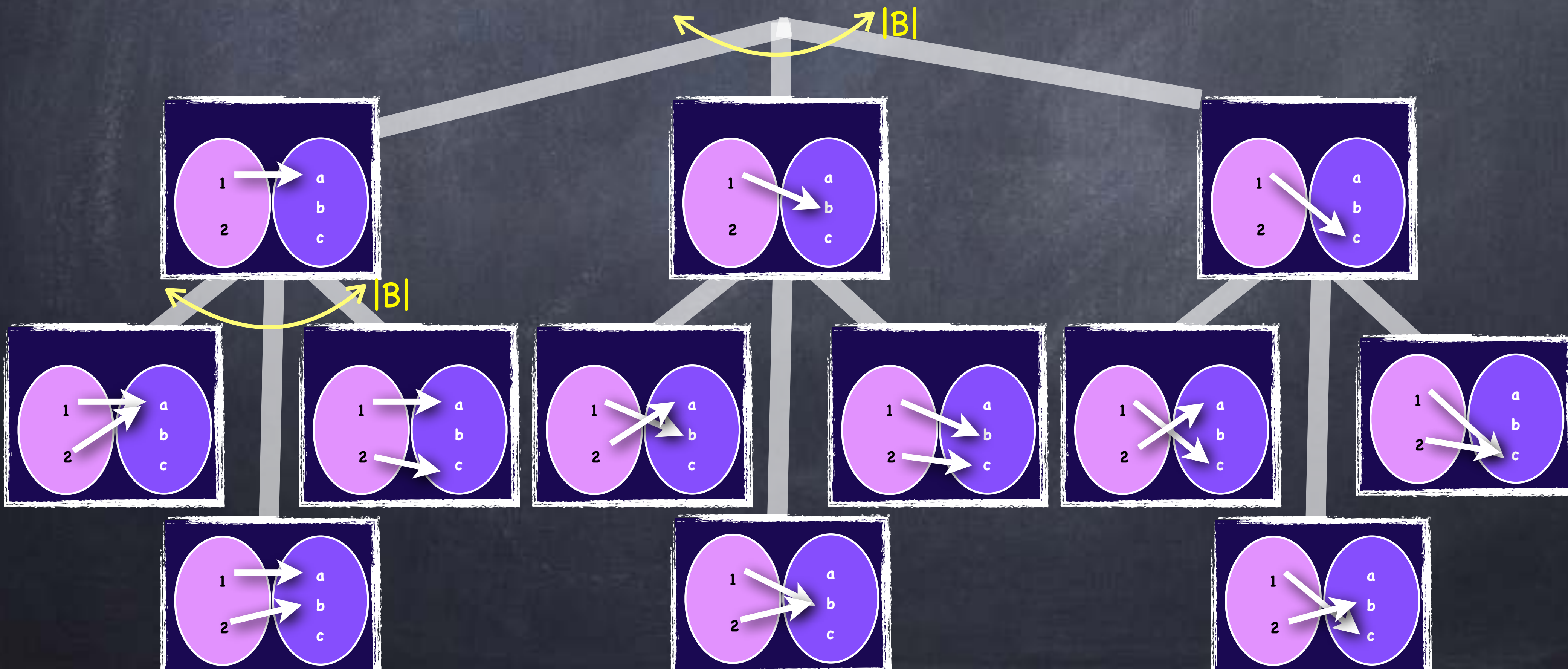
Counting

Lecture 11

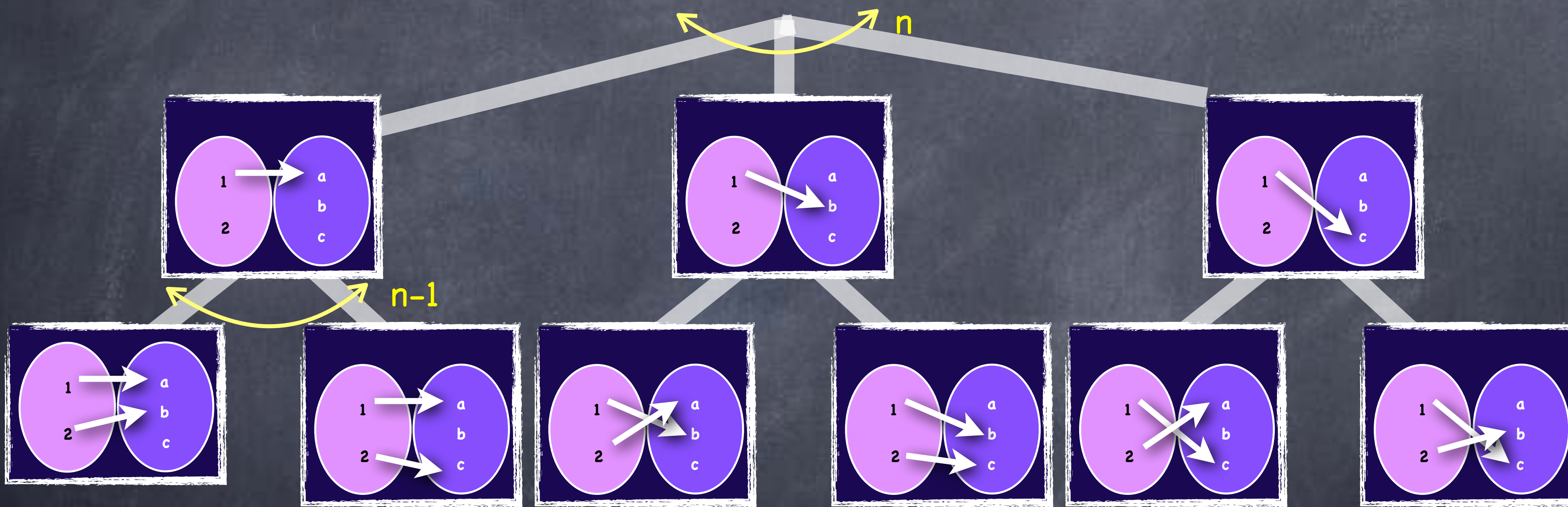
How many Functions?

- # functions from A to B (finite): $|B|^{|A|} = \underbrace{|B| \cdot |B| \cdot \ldots \cdot |B|}_{|A| \text{ times}}$
- e.g. A={1,2} B={a,b,c}



How many One-to-One Functions?

- \# <u>one-to-one</u> functions from A to B, where |A|=k and |B|=n



- $\underbrace{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}_{\text{k times}} = n!\ /\ (n-k)!$

P(n,k)

a.k.a. falling factorial, $(n)_k$

How many Functions?

- Suppose $|A|=k$, $|B|=n$
  - Say, $A=[k]\triangleq\{1,\ldots,k\}$ and $B=[n]\triangleq\{1,\ldots,n\}$
- # functions from A to B: n^k
- # one-to-one functions from A to B (assuming $n\geq k$) is $P(n,k) = n\cdot(n-1)\cdot\ldots\cdot(n-k+1) = n! / (n-k)!$ [Note: $0! = 1$]
  - Pigeonhole Principle: There is a one-to-one function from A to B only if $|B|\geq|A|$. $P(n,k) = 0$ for $k>n$
- # bijections from A to B (only if $|A|=|B|$) is $P(n,n) = n!$
- # onto functions? A little more complicated. (Later)

e.g., number of ternary logical operators, op : $\{T,F\}\times\{T,F\}\times\{T,F\}\rightarrow\{T,F\}$ is 2^8

Permutations

- Permutations refer to arrangements of, say, symbols in a string
  - abcde, cadeb, cabed, ...
  - A permutation of a set of symbols could be thought of as a function that assigns to each position a symbol (without repeating symbols)

Bijection

1	2	3	4	5
c	a	d	e	b

  - Sometimes want to consider shorter strings obtained from the given string (without repeating symbols)

One-to-one

1	2	3
c	a	d

P(5,3) ways

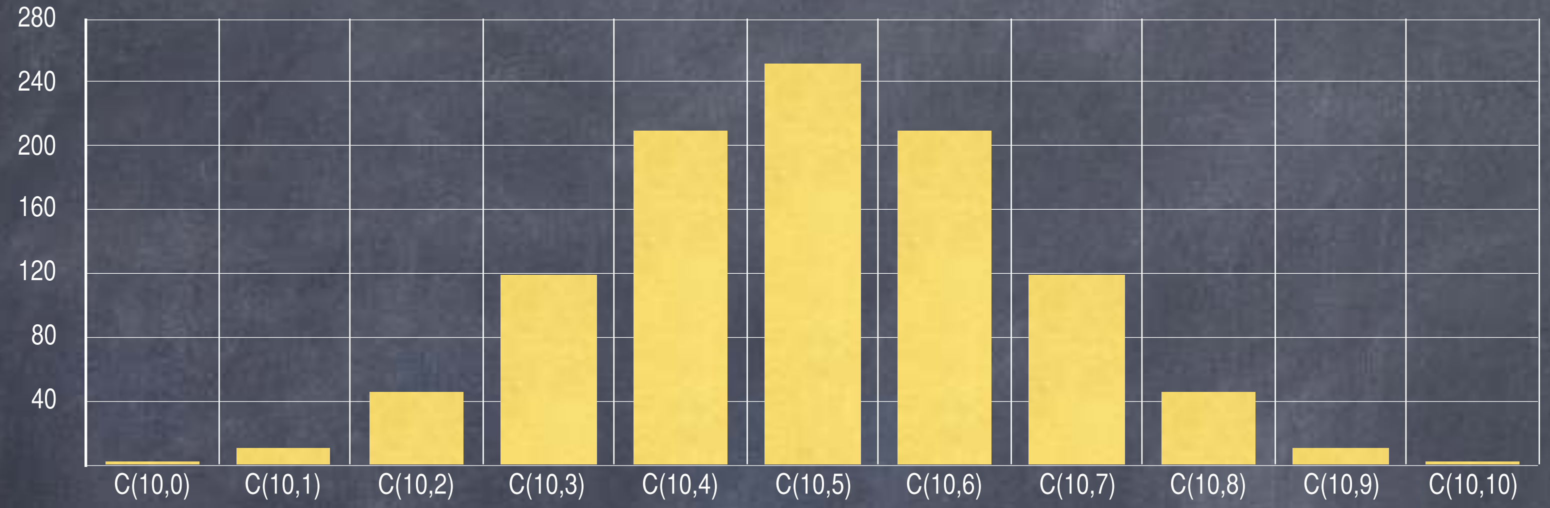
Combinations

- We can represent subsets as strings without repetitions
- e.g., $\{a,c,d\} \subseteq \{a,b,c,d,e\}$ can be represented as <u>acd</u>
- But the same subset can be represented in multiple ways: <u>adc</u>, <u>cad</u>, ...
  - We know exactly how many ways
  - k! ways to write a subset of size k
- \# k-symbol subsets of n-symbol alphabet
  = # repetition-free strings of length k, divided by k!
  - $C(n,k) = P(n,k)/k! = n! / ((n-k)! \cdot k!)$

Also written $\binom{n}{k}$

C(n,k)

- For $n,k \in \mathbb{N}$, $C(n,k) = n!/(k!(n-k)!)$ if $k \leq n$, and 0 otherwise



- $C(n,k) = C(n,n-k)$
  - Selecting k out of n elements is the same as unselecting n-k out of n elements
- $C(n,0) = C(n,n) = 1$
  - In particular, $C(0,0) = 1$
    (how many subsets of size 0 does ∅ have?)
- $C(n,0) + C(n,1) + \ldots + C(n,n-1) + C(n,n) = 2^n$

C(n,k)

- <u>$C(n,k)$ is the coefficient of x^k in the expansion of $(1+x)^n$</u>
- Fully expanding $(1+x)^n$ results in 2^n terms
- $(1+x)\cdot(1+x)\cdot(1+x) = (1+x)\cdot(1\cdot 1 + 1\cdot x + x\cdot 1 + x\cdot x)$

$$= 1\cdot 1\cdot 1 + 1\cdot 1\cdot x + 1\cdot x\cdot 1 + 1\cdot x\cdot x$$

$$+ x\cdot 1\cdot 1 + x\cdot 1\cdot x + x\cdot x\cdot 1 + x\cdot x\cdot x$$

  - Each term is of the form $?\cdot ?\cdot ?$ where each ? is 1 or x
- Coefficient of x^k = number of strings with exactly k x's out of the n positions = $C(n,k)$
- Note: coefficient of x^k in $(1+x)\cdot(\ldots + ax^{k-1} + bx^k + \ldots)$ is $a+b$
  - a = coefficient of x^{k-1} in $(1+x)^{n-1} = C(n-1,k-1)$
    b = coefficient of x^k in $(1+x)^{n-1} = C(n-1,k)$

C(n,k)

- <u>$C(n,k) = C(n-1,k-1) + C(n-1,k)$</u> (where $n,k \geq 1$)
  - Easy derivation: Let $|S|=n$ and $a \in S$.
    $C(n,k)$ = # k-sized subsets of S containing a
    + # k-sized subsets of S not containing a
- In fact, gives a <u>recursive definition</u> of $C(n,k)$
  - <u>Base case</u> (to define for $k \leq n$):
    $C(n,0) = C(n,n) = 1$ for all $n \in \mathbb{N}$
  - Or, to define it for all $(n,k) \in \mathbb{N} \times \mathbb{N}$
    <u>Base case:</u> $C(n,0)=1$, for all $n \in \mathbb{N}$,
    and $C(0,k)=0$ for all $k \in \mathbb{Z}^+$

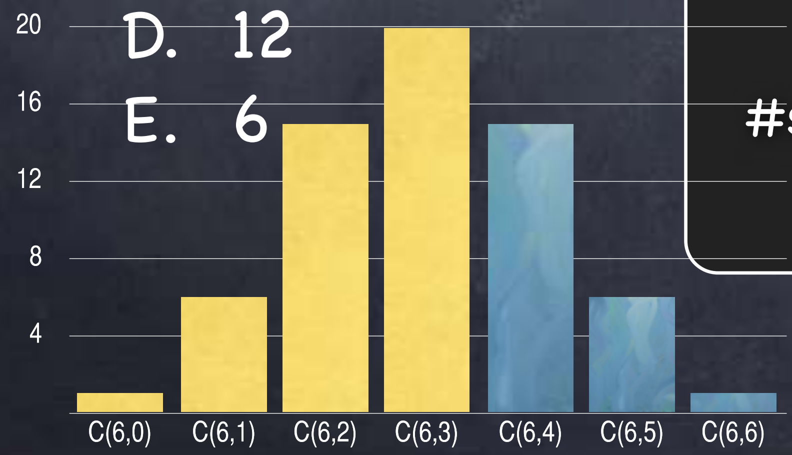
n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	2	1	0	0	0	0
3	1	3	3	1	0	0	0
4	1	4	6	4	1	0	0
5	1	5	10	10	5	1	0
6	1	6	15	20	15	6	1

Question

02:00

- How many bit strings of length 6 are there with more 1s than 0s?

A. 32
B. 22
C. 15
D. 12
E. 6



#strings with 0 zero = C(6,0) = 1
#strings with 1 zero = C(6,1) = 6
#strings with 2 zeros = C(6,2) = 15

#strings = 2^6 = 64
#strings with equal # 0s & 1s = C(6,3) = 20
Half of remaining strings = 44/2 = 22

Counting and Discounting

Ordered Bell Number: Number of weak orderings

- How many ways can 4 horses finish a race, if ties are possible?
  - Only one position (all first): 1 way
  - Two positions, 1 & 2: Assign each horse 1 or 2 (2^4 ways), but discount ones which use only one number (2 ways): 14 ways
  - Three positions, 1, 2 & 3: Assign each horse 1, 2 or 3, but discount ones which use only 2 numbers, and ones which use only one number: $3^4 - C(3,2) \cdot 14 - C(3,1) \cdot 1 = 81-45=36$
  - All 4 positions: $4!\ (=4^4-4 \cdot 36-6 \cdot 14-4 \cdot 1 = 256-232) = 24$
  - Total = $1+14+36+24=75$
- 5 horses?
  - $1+(2^5-2)+(3^5-3 \cdot 30-3 \cdot 1)+(4^5-4 \cdot 150-6 \cdot 30-4 \cdot 1)+5!=541$

Counting Onto Functions

- How many onto functions from A to B, if $|A|=k$, $|B|=n$?
  - Let's call it $N(k,n)$

$n^k - n\,(n-1)^k + C(n,2)\,(n-2)^k - \ldots$

- Claim: **$N(k,n) = \Sigma_{i=0 \text{ to } n} (-1)^i\, C(n,i)\, (n-i)^k$**
- $N(k,n) = |\{f:A\to B\}| - |\cup_i \{f:A\to S_i\}|$ where $S_i = B - \{i\}$
  - Inclusion-exclusion to count $|\cup_i \{f:A\to S_i\}|$
    - $\{f:A\to S_{i1}\} \cap\ldots\cap \{f:A\to S_{it}\} = \{f:A\to S_{i1} \cap \ldots \cap S_{it}\}$
    - $|S_{i1} \cap \ldots \cap S_{it}| = n-t$. So, $|\{f:A\to S_{i1} \cap \ldots \cap S_{it}\}| = (n-t)^k$
    - Number of combinations of the form $\{i_1,\ldots,i_t\} = C(n,t)$
    - $|\cup_i \{f:A\to S_i\}| = n\,(n-1)^k - C(n,2)\,(n-2)^k + C(n,3)\,(n-3)^k - \ldots$
- $N(k,n) = n^k - \Sigma_{i=1 \text{ to } n} (-1)^{i+1}\, C(n,i)\, (n-i)^k = \Sigma_{i=0 \text{ to } n} (-1)^i\, C(n,i)\, (n-i)^k$

Counting Partitions

- Recall: $\{P_1,\ldots,P_d\}$ is a partition of A if $A = P_1 \cup \ldots \cup P_d$, for all distinct i,j, $P_i \cap P_j = \emptyset$, and no P_i is empty
- How many partitions does a set A of k elements have?
  - S(k,n): #ways can A be partitioned into exactly n parts

    Stirling number of the 2nd kind
    - Suppose we labeled the parts as 1,…,n
    - Such a partition is simply an onto function from A to [n]
      - N(k,n) ways
    - But in a partition, the parts are not labelled. With labelling, each partition was counted n! times.
  - S(k,n) = N(k,n)/n!

Balls and Bins

- How many ways can I throw k labelled balls into n labelled bins?
  - Number of functions $f:A \to B$, where $|A|=k$, $|B|=n$ (n^k)
- How many ways can I throw k labelled balls into n labelled bins so that no bin is empty?
  - Number of onto functions, $N(k,n)$
- How many ways can I throw k labelled balls into n <u>unlabelled</u> bins so that no bin is empty?
  - Number of partitions, $S(k,n)$
- How many ways can I throw k <u>unlabelled</u> items (balls) into n <u>labelled</u> bins?

Combinations With Repetitions

- How many ways can I throw k (indistinguishable) balls into n (distinguishable) bins?
- A multi-set (a.k.a "bag") is like a set, but allows an element in it to occur one or more times
  - Order doesn't matter
  - e.g., [a, a, b, d] = [b, a, d, a]
- How many multi-sets of size k are there, with elements coming from a set of size n?
  - e.g., how many ways can I place orders for 10 books from a catalog of 20 books (I may order multiple copies of the same book)?
  - Balls and bins: bins are the catalog titles, and balls are my orders

Combinations With Repetitions

- How many ways can I throw k (indistinguishable) balls into n (distinguishable) bins?
- Each such combination can be represented using n-1 "bars" interspersed with k "stars"
  - e.g., 3 bins, 7 balls: ★ ★ ★ | ★ ★ ★ | ★
    Or, | | ★ ★ ★ ★ ★ ★ ★ (first two bins are empty)
  - Number of such combinations = ?
    - (n-1)+k places. Choose n-1 places for bars, rest get stars
    - C (n+k-1, k) ways

Combinations With Repetitions

- How many solutions are there for the equation $x+y+z = 11$, with $x,y,z \in \mathbb{Z}^+$?
- 3 bins, 11 balls: But no bin should be empty!
- First, throw one ball into each bin
- Now, how many ways to throw the remaining balls into 3 bins?
  - 3 bins, 8 balls
  - 2 bars and 8 stars: e.g., ★ | | ★ ★ ★ ★ ★ ★ ★
  - C(10,2) solutions
    - e.g., above distribution corresponds to $x=2$, $y=1$, $z=8$

Summary

- Number of functions $f:A \rightarrow B$, where $|A|=k$, $|B|=n$ (k labelled balls into n labelled bins): n^k
- Number of one-to-one functions (k labelled balls into n labelled bins, so that no bin has two balls): $P(n,k)$
- Number of combinations of n items from A (k unlabelled balls into n labelled bins, so that no bin has two balls): $C(n,k)$
- Number of onto functions (k labelled balls into n labelled bins, so that no bin is empty): $N(k,n)$
- Number of partitions of A into n parts (k labelled balls into n unlabelled bins, so that no bin is empty): $S(k,n)$
- Number of multiset subsets of B of size k (k unlabelled balls into n labelled bins): $C(n+k-1,k)$
- How many ways can I throw k <u>unlabelled</u> items (balls) into n <u>unlabelled</u> bins? Partition function, $p_k(n)$ (complicated!)