Mid-Semester Wrap Up

Lecture 12

Topics to be covered

Recursion

Bounding
big-O

Trees

Induction

Counting

Numbers and patterns therein

Graphs

Basic tools for expressing ideas
Logic, Proofs,
Sets, Relations, Functions

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Pigeon Hole Principle Examples

- n people went to a Chinese restaurant, and sat on a large circular table, with a rotating disk in the centre
- Each one ordered a different dish. The servers brought all the dishes out at the same time and placed them on the disk, one in front of each person
- It turned out that no person had their dish in front of them
- Prove that they can rotate the disk in such a way that at least two diners will have their dishes in front of them
- For each person i, calculate d_i the number of positions that the disk needs to be rotated clockwise to get to their dish
- $d_i \in \{1,\ldots,n-1\}$. $d_1, \ldots, d_n$. So at least two values d_i, d_j which are equal. Rotate the disk by that much.

Pigeon Hole Principle Examples

- Given an arbitrary sequence of n integers $a_1, \ldots, a_n$, there is some sequence of consecutive elements $a_i, a_{i+1}, \ldots, a_{i+d}$ which sums up to a multiple of n

  [e.g., $a_1=4$, $a_2=2$, $a_3=7$, $a_4=4$, $a_5=2$]

- First consider all n sequences starting with a_1, and let their sums be $s_1, \ldots, s_n$. [e.g., $s_1=4$, $s_2=6$, $s_3=13$, $s_4=17$, $s_5=19$]
- If any one of them is a multiple of n, we are done
- Otherwise, $r_i = \text{rem}(s_i,n)$ is in the range [1,n-1] for each i, and so by PHP, there are i,j s.t. i<j (w.l.o.g.) and $r_i \equiv r_j \pmod{n}$

  [e.g., $s_1=4$, $s_5=19$]

- Then, $a_i + \ldots + a_j = s_j - s_i$ is a multiple of n [2+7+4+2 = 15]

Pigeon Hole Principle Examples

- A generalisation of PHP: If n numbers $x_1,\ldots,x_n$ add up to y, then there is an i s.t. $x_i \geq y/n$
  - Because if all $x_i < y/n$, then $x_1 + \ldots + x_n < n\,(y/n) = y$
  - "Everyone cannot be below average."
    - Also true that $\exists i\; x_i \leq y/n$
  - For x_i being integers, can state as $\exists i\; x_i \geq \lceil y/n \rceil$
    - E.g., if $y=n(r-1)+1$, $\exists i\; x_i \geq r$
- Further generalisation: If $x_1+\ldots+x_n \geq r_1+\ldots+r_n$, $\exists i$ s.t. $x_i \geq r_i$

Numbers

GCD

a_i	b_i	q_i	u_i	v_i
20	9	2	0	1
9	2	4	1	-2
2	1	2	-4	9
1	0			

- Euclidean algorithm to find gcd(a,b)
  - $(a_0,b_0) \leftarrow (a,b)$ where $a \geq b$
  - for ($i=0$; $b_i > 0$; i++)
    - $a_{i+1} \leftarrow b_i$; $b_{i+1} \leftarrow \text{rem}(a_i,b_i)$;
  - return a_i
- Extended Euclidean algorithm to find u,v s.t. au+bv=gcd(a,b)
  - Idea: keep track of u_i,v_i s.t. $b_i = u_i a + v_i b$.
    Let $q_i = \lfloor a_i/b_i \rfloor$ so that $b_{i+1} = a_i - q_i b_i$
    - $u_0=0$, $v_0=1$. $u_1=1$, $v_1=-q_0$ (because $b_1 = a - q_0 b$)
    - For $i \geq 1$, recall $a_i = b_{i-1} = u_{i-1}a + v_{i-1}b$.
      So, $b_{i+1} = a_i - q_i b_i = (u_{i-1}a + v_{i-1}b) - q_i(u_i a + v_i b)$
      - $u_{i+1} = u_{i-1} - q_i u_i$. $v_{i+1} = v_{i-1} - q_i v_i$.

Numbers
Totient Function

- $\varphi(m) = |\mathbb{Z}_m^*|$, where $\mathbb{Z}_m^* = \{\ a \mid \gcd(a,m) = 1\ \}$
  - e.g., for p prime, $\varphi(p) = p-1$
- $\forall a \in \mathbb{Z}_m^*,\quad a^{\varphi(m)} \equiv 1 \pmod{m}$
- $2^{2019} \pmod{100} = ?$
  - $2 \notin \mathbb{Z}_{100}^*$, so can't use Euler's theorem directly
  - $2^{2019} \equiv 2^{\mathrm{rem}(2019,\varphi(25))} \pmod{25}$

    $\varphi(25) = 5(5-1) = 20$. And, $2019 \equiv -1 \pmod{20}$.

    $2^{2019} \equiv 2^{-1} \equiv 13 \pmod{25}$.
    - $2^{-1} \pmod{m}$ for odd m is $(m+1)/2$
  - Solve for x s.t. $x \equiv 13 \pmod{25}$; $x \equiv 0 \pmod{4}$.
    - Multiple of 4 in 13,38,63,88 = 88

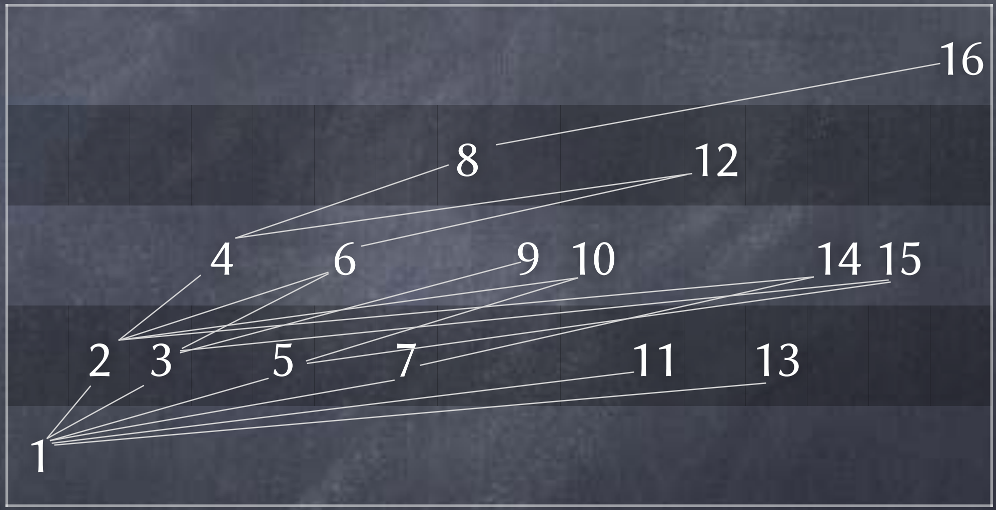
Numbers
Generators of $\mathbb{Z}_p^*$

- Important Fact (won't prove): If p is a prime, then $\exists g\in\mathbb{Z}_p^*\ \forall x\in\mathbb{Z}_p^*\ \exists k,\ 0 \leq k < p-1,\ \ x=g^k$
  - Such a g is called a "generator of $\mathbb{Z}_p^*$"
- Consequence: Can analyse $\mathbb{Z}_p^*$ by considering it as $\{1,g,g^2,\ldots,g^{p-2}\}$, with exponents from $\mathbb{Z}_{p-1}$ (because $g^{p-1} = 1$)
- e.g., How many generators?
  - Fix one generator g. Elements of $\mathbb{Z}_p^*$ are $1,g,g^2,\ldots,g^{p-2}$
  - g^t is a generator iff $\{1,g,g^2,\ldots,g^{p-2}\} = \{1,g^t,g^{2t},\ldots,g^{(p-2)t}\}$ (mod p)
    - i.e., $\{0,1,\ldots,p-2\} = \{0,t,\ldots,(p-2)t\}$ (mod p-1)
    - i.e., $\gcd(t,p-1) = 1$
    - $\varphi(p-1)$ such t
  - Number of generators of $\mathbb{Z}_p^* = \varphi(p-1)$

Poset

- Partial order: Reflexive, Anti-Symmetric, Transitive
  - e.g., Divisibility poset $(\mathbb{Z}^+, \preccurlyeq)$: $a \preccurlyeq b$ iff $a|b$
- a covered by b $(a \lessdot b)$ iff $a \preccurlyeq b$, $a \neq b$, and there is no x s.t. $a \preccurlyeq x \preccurlyeq b$

Why?



- $\preccurlyeq$ is the reflexive and transitive closure of $\lessdot$
- Hasse diagram: the graph for $\lessdot$ (with edges pointing upwards, so that the arrow is implicit)

Lattices

- A lattice is a poset in which every pair of elements {a,b} has a greatest lower bound and a least upper bound
  - Implies the same for every finite set of elements
  - g.l.b.({a,b,c}) = g.l.b.({g.l.b.({a,b}), c})
    - Let z = g.l.b.({a,b}) (which exists).
      $x \in$ LowerBd({a,b,c}) $\Rightarrow$ $x \in$ LowerBd({a,b}) $\Rightarrow$ $x \preccurlyeq z$;
      so $x \in$ LowerBd({z,c}).
      Conversely, if $x \in$ LowerBd({z,c}), $x \preccurlyeq z \preccurlyeq a$, $x \preccurlyeq z \preccurlyeq b$, $x \preccurlyeq c$.
      i.e., LowerBd({a,b,c}) = LowerBd({z,c}).
      But RHS has a greatest element. So LHS does too: g.l.b.({a,b,c})
- e.g., Divisibility poset (g.l.b. is g.c.d., l.u.b. is l.c.m.), subsets poset (g.l.b. is intersection, l.u.b. is union)

Chains

- Consider poset $(S, \preccurlyeq)$
- $C \subseteq S$ is called a chain if $\forall$ $a,b \in C$, either $a \preccurlyeq b$ or $b \preccurlyeq a$
- $\text{height}(a) = \max_{C \text{ chain with } a \text{ as maximum}} |C|$
- Height of a poset = $\max_{C \text{ chain}} |C|$
  - Height of poset = $\max_{a \in S} \text{height}(a)$
- A <u>maximal chain</u> is a chain which is not contained in a longer chain
  - Suppose C is a finite maximal chain. The greatest element of C exists and is a maximal element of the poset. The least element of C exists and is a minimal element of the poset

Mirsky's Theorem

- $A \subseteq S$ is called an anti-chain if $\forall a,b \in A$, $a \preccurlyeq b \longrightarrow a=b$
- **Given a poset of height H, the least number of anti-chains needed to partition S is H**
  - Need at least H anti-chains to partition S
    - Consider a chain of length H (exists). No two elements in that chain can be in one anti-chain
  - And need no more: use $\{ A_h \mid h=1,\ldots,H \}$, where $A_h \triangleq \{ a \mid \text{height}(a)=h \}$
    - A_h is an anti-chain — Why?
    - $\{ A_h \mid h=1,\ldots,H \}$ is a partition — Why?

Mirsky's Theorem: Example

- Given a poset of height H, the least number of anti-chains needed to partition S is H
- Claim: Any poset with n elements must have either (i) a chain and an anti-chain both of size equal to $\sqrt{n}$ or (ii) a chain or an anti-chain of size greater than $\sqrt{n}$
- Suppose (ii) doesn't hold. All chains, anti-chains of size $\leq \sqrt{n}$.
  So, height of poset, $H \leq \sqrt{n}$.
  By Mirsky, can partition the poset into H anti-chains, say of sizes $n_1,\ldots,n_H$. So, $n = n_1 + \ldots + n_H$.
  But $n_i \leq \sqrt{n}$. So $n \leq H\sqrt{n}$. Thus $H=\sqrt{n}$. Then, each $n_i = \sqrt{n}$.
  So (i) holds.

Mirsky's Theorem: Example

- Consider the numbers from 1 to n arranged in an arbitrary order on a line. There must exist a $\sqrt{n}$-long subsequence of that is completely increasing or completely decreasing as you move from right to left.
  - Consider poset $([n],\preccurlyeq)$ defined as: $a \preccurlyeq b$ if $a \leq b$ and a appears not "later than" (left of) b in the line
    - Verify poset!
  - Chain: An increasing subsequence (right to left)
  - Anti-chain: If a appears before b, but not $a\preccurlyeq b$, then $a>b$. So a decreasing sequence (right to left)
  - Previous claim: Chain or anti-chain of length at least $\sqrt{n}$

Bijections: Infinite sets

- e.g., f: $\mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$
    - f is a bijection
- A set S is countably infinite if there is a bijection from S to $\mathbb{Z}$
- e.g., set of even integers $\mathbb{E}$. f: $\mathbb{E} \to \mathbb{Z}$ where $f(x)=x/2$, is a bijection
    - "Two countably infinite sets are only as numerous as one"
- e.g., there is a bijection f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ How?
    - "Countably infinitely many countably infinite sets are only as numerous as one"
- (We will return to this later)