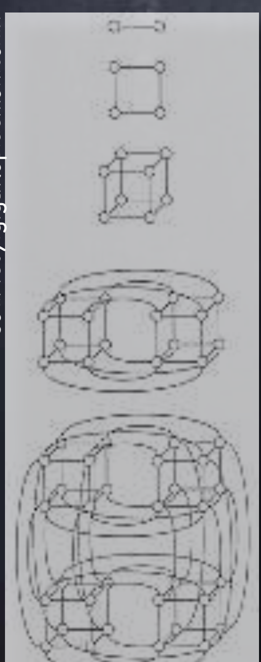
Graphs

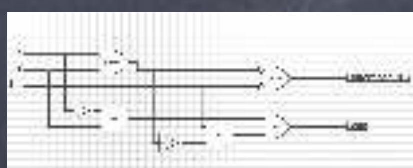
Lecture 13

Graphs

- What is "connected" to what
- Many things we deal with in computer science are graphs
  - Networks: humans, communication, computation, transportation, knowledge

Courtesy:gigaflop.demon.co.uk





Courtesy: Microsoft Academic Search

Courtesy: cablemap.info

Courtesy: New Scientist

Courtesy: Digital Humanities Specialist @ Stanford

Simple Graphs

- A simple graph $G = (V,E)$, where
  - $E \subseteq \{ \{a,b\} \mid a,b \in V, a \neq b \}$
- V is the set of nodes E the set of edges
- V non-empty and finite (for us)
- Note: the "drawing" is not part of the graph, only the connectivity is

Simple Graphs

- Recall graphs for relations: directed graphs with self-loops
  - Each element in the domain forms a node
  - Each ordered pair (a,b) in the relation forms an edge
    - Edges of the form (a,a) are "self-loops"
- A simple graph is a symmetric, irreflexive relation
  - Symmetric: An undirected edge {a,b} can be modelled as two directed edges (a,b) and (b,a)
  - Irreflexive: No self-loops
- In a "non-simple" graph, can allow more than one edge between any pair (multigraphs), or more generally, allow weights on edges (weighted graphs)

Examples

- Complete graph K_n : n nodes, with all possible edges between them
  - $E = \{ \{a,b\} \mid a,b \in V, a \neq b \}$
  - # edges, $|E| = n(n-1)/2$

- Cycle C_n : $V = \{ v_1,\ldots,v_n \}$, $E = \{ \{v_i,v_j\} \mid j=i+1 \text{ or } (i=1 \text{ and } j=n) \}$

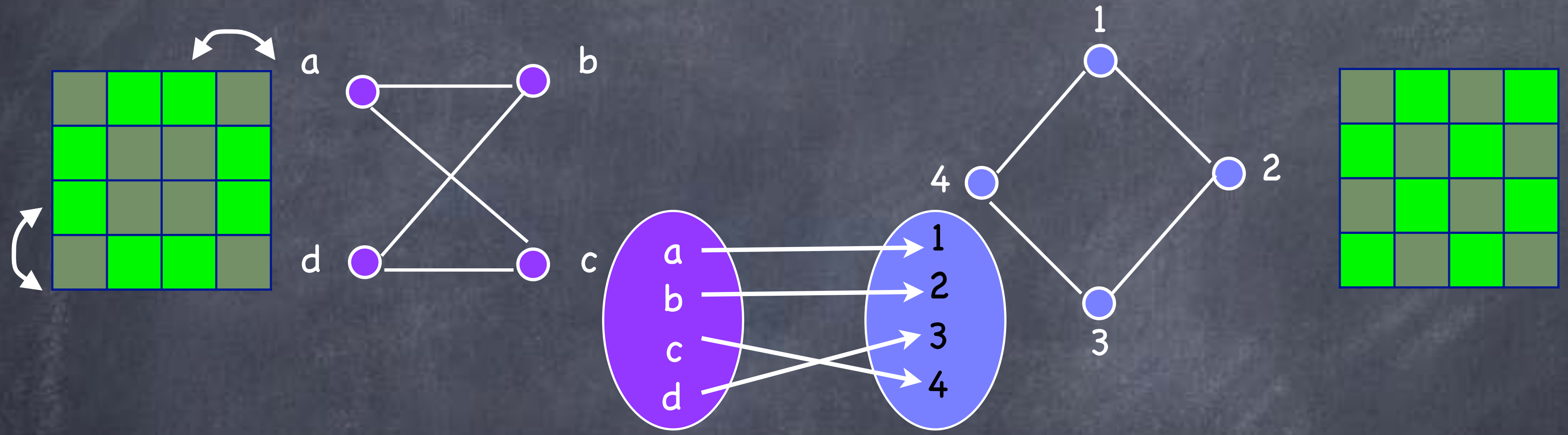
- Bipartite graph : $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ (i.e., a partition), and no edge between two nodes in the same "part": $E \subseteq \{ \{a,b\} \mid a \in V_1, b \in V_2 \}$
  - e.g., C_n where n is even

- Complete bipartite graph $K_{n1,n2}$: Bipartite graph, with $|V_1|=n_1$, $|V_2|=n_2$ and $E = \{ \{a,b\} \mid a \in V_1, b \in V_2 \}$
  - # edges, $|E| = n_1 \cdot n_2$

- Later: Hypercube, Trees

Graph Isomorphism

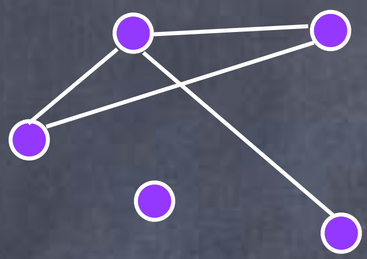
- $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are isomorphic if there is a bijection $f:V_1 \rightarrow V_2$ such that $\{u,v\} \in E_1$ iff $\{f(u),f(v)\} \in E_2$



- Computational problem: check if two graphs (given as adjacency matrices) are isomorphic
  - Can rule out if certain "invariants" are not preserved (e.g. $|V|,|E|$)
- In general, no "efficient" algorithm known, when graph is large
  - Some believe no efficient algorithm exists!

Degree of a node

- Given a simple graph $G = (V,E)$, for each node $v \in V$, the degree of v is the number of edges incident on v



- Formally, $\deg(v) = |\ \{\ u : \{u,v\} \in E\ \}\ |$

Note: Definition restricted to simple graphs

- Counting edges in two different ways: $2 \cdot |E| = \Sigma_{v \in V} \deg(v)$
- Degree sequence: sorted list of degrees. (e.g.: 0,1,2,2,3)
- Degree sequence invariant under isomorphism

ZBVY

Question

02:00

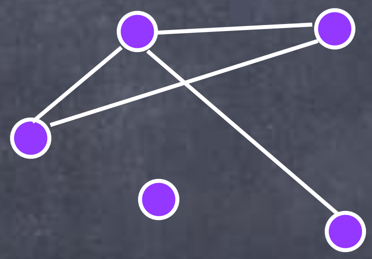
- A graph is said to be d-regular if all nodes have degree d. How many edges are there in a 3-regular graph with 6 nodes?

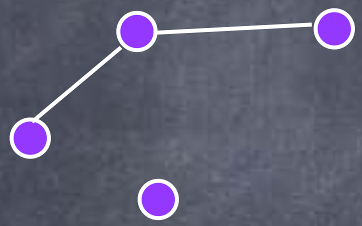
A. Such a graph doesn't exist
B. 18
C. 12
D. 9
E. It depends on the graph

e.g., $K_{3,3}$
$n \cdot d/2$ (unless n & d odd)

Subgraphs

- A subgraph of G = (V,E) is a graph G′ = (V′,E′) such that $V' \subseteq V$ and $E' \subseteq E$





- To get a subgraph: Remove zero or more vertices, remove all edges incident on them, and further remove zero or more edges
    - Induced subgraph: omit the last step

Walks, Paths & Cycles

- A walk (of length k, $k \geq 0$) from node a to node b is a sequence of nodes $(v_0, v_1, \ldots, v_k)$ such that
  - $v_0 = a$, $v_k = b$
  - for all $i \in \{0,\ldots,k-1\}$, the edge $\{v_i,v_{i+1}\} \in E$
- Length is the number of edges in a walk. Could be 0.
- If a walk has no node repeating, then it is called a path
- If a walk of length $k \geq 3$ has $v_0=v_k$, but no other two nodes are equal, then it is called a cycle
  - Note: we require a cycle to be of length at least 3
- A graph is acyclic if it has no cycles (i.e., no C_k is a subgraph of G)

Connectivity

- Given a graph G, whether there is a path between two nodes u and v is an important question regarding G
  - u is said to be connected to v if there is such a path
  - Prove: u connected to v iff there is a walk from u to v
- Relation Connected(u,v) is an equivalence relation
  - Reflexive, Symmetric and Transitive

    Walks can be spliced together to get walks
  - Equivalence classes of this relation are called the connected components of G

Distance

Prove via contradiction

Hence ∃ walk → ∃ path

In many applications, the edges on the graph will have "lengths". For us, typically all edges are of length 1.

- Shortest walk between nodes u and v is always a path
- Shortest path is of great interest in many applications
  - e.g., nodes correspond to locations on a map and edges are roads, optic fibers etc.
  - Also, graph can be used to model probabilistic processes, with shortest path indicating the most likely outcome
- Length of the shortest path between u and v is called the distance between u and v (∞ if no path)

$$\min_{W:\ u\text{-}v\ \text{walk}} \mathrm{Length}(W)$$

- Diameter is the largest distance in a graph (can be ∞)

$$\max_{u,v} \mathrm{Distance}(u,v) = \max_{u,v} \min_{W:\ u\text{-}v\ \text{walk}} \mathrm{Length}(W)$$