Graphs

Lecture 15

Matchings

- A matching in a graph G=(V,E) is a set of edges which do not share any vertex
  - i.e., a set $M \subseteq E$ s.t. $\forall e_1,e_2 \in M$, $e_1 \neq e_2 \rightarrow e_1 \cap e_2 = \emptyset$
- Trivial matchings: $\emptyset$ is a valid matching. For any $e \in E$, $\{e\}$ is a valid matching, too.
- A perfect matching: All nodes are matched by M.
  - i.e., $\forall v \in V$, $\exists$ $e \in M$ s.t. $v \in e$
  - May or may not exist
- Algorithmic task: given a graph find a largest (maximum) matching
  - Efficient algorithms do exist (we will not cover them here)
- Why do we care?

Graphs in action

Graph Matching in Action

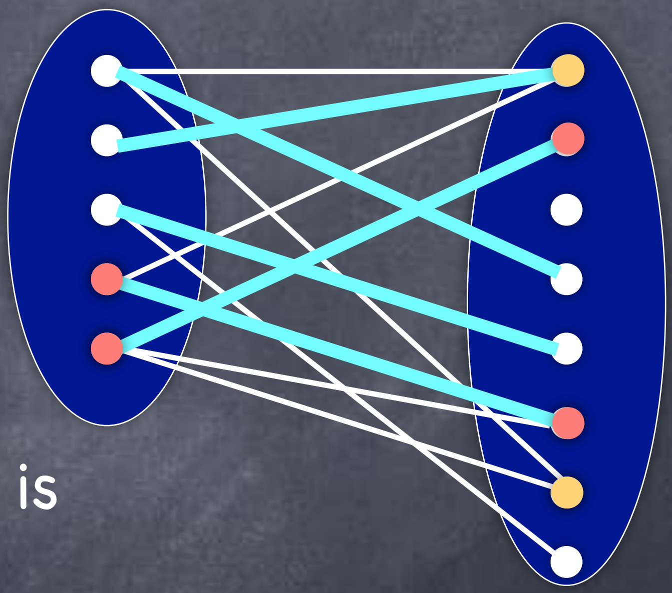
- Matching in bipartite graphs
  - Assigning tasks: Workers and tasks are the nodes. A worker is connected to a task by an edge if the worker is qualified for the task. A worker can perform only one task, and each task needs only one worker.
    - Maximum matching: Getting most tasks assigned to workers
  - Advertisements and slots (e.g., on webpages): each advertiser specifies which slots they prefer; the goal is to maximise the number of slots filled
  - Additional issues: weights (maximum weight matching), costs (e.g., minimum cost perfect matching), "online matching"

Shrinking Neighbourhood in Bipartite Graphs

- Given a graph $G = (V,E)$, and $v \in V$, we define v's neighbourhood:
  - $\Gamma(\{v\}) \triangleq \{ u \mid \{u,v\} \in E \}$
- More generally, neighbourhood of a set $S \subseteq V$:
  - $\Gamma(S) \triangleq \bigcup_{v \in S} \Gamma(\{v\})$
- Recall: A bipartite graph, $G=(X,Y,E)$ is a graph $(X \cup Y,E)$ where $X \cap Y = \emptyset$ and, $\forall e \in E$, $|e \cap X| = |e \cap Y| = 1$
  - Note that for $S \subseteq X$, we have $\Gamma(S) \subseteq Y$
  - For $S \subseteq X$, we shall say S is <u>shrinking</u> $|\Gamma(S)| < |S|$
  - More generally, for $B \subseteq Y$, S <u>shrinking in B</u> if $|\Gamma(S) \cap B| < |S|$
    - i.e., the set of neighbours of S in B is smaller than S

Complete Matchings and Shrinking Neighbourhoods

- Given bipartite G=(X,Y,E), a complete matching from X to Y is a matching M s.t. |M|=|X|
  - Exists only when |X| ≤ |Y| because |M| ≤ min (|X|,|Y|)
- If |X|=|Y|, a complete matching from X to Y is also a complete matching from Y to X
  - And is a perfect matching
- If there is a complete matching from X to Y, then ∀S ⊆ X, S is not shrinking in Y [Why?]



Hall's Theorem

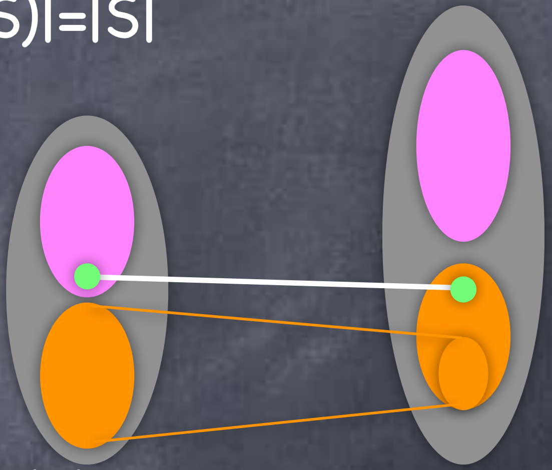
- **Bipartite graph G=(X,Y,E) has a complete matching from X to Y iff no subset of X is shrinking**
- We saw that Hall's condition is necessary for a complete matching from X to Y
- Proof of sufficiency by strong induction on |X|.

Hall's Theorem

- Claim: No shrinking $S \subseteq X \Rightarrow \exists$ a complete matching from X into Y
- Proof by strong induction on |X|.
- Base case, |X|=1: ✓ (How?)
- Induction step: Suppose claim holds for graphs with $|X| \leq k$.
  - Given graph (X,Y,E) with |X|=k+1, s.t. $\forall U \subseteq X$, $|\Gamma(U)| \geq |U|$
  - Pick an arbitrary $x \in X$, and an arbitrary neighbour y of x (since {x} is not shrinking, x has a neighbour).
  - Case 1: There is a complete matching from X–{x} to Y–{y}.
    Then, X has a complete matching into Y ✓
  - Case 2: No complete matching from X–{x} to Y–{y}.

Hall's Theorem

- Case 2: No complete matching from X-{x} to Y-{y}
- By ind. hyp., ∃ S ⊆ X-{x} s.t. S is shrinking in Y-{y}
  - But S not shrinking in Y. So, y∈Γ(S) and |Γ(S)|=|S|
- Claim: ∃ a complete matching from S into Γ(S)
  - |S| ≤ k, and no subset of S is shrinking. So by ind. hyp. ∃ a complete matching of S into Y. This must be into Γ(S)



- Claim: ∃ a complete matching from X-S into Y-Γ(S)
  - |X-S|≤k. Enough to show ∀T⊆X-S, |Γ(T)∩(Y-Γ(S))| ≥ |T|
  - Γ(T)∩(Y-Γ(S)) = Γ(T)-Γ(S) = Γ(U)-Γ(S), where U=T∪S. |Γ(U)-Γ(S)| ≥ |Γ(U)|-|Γ(S)| = |Γ(U)|-|S|. But, |Γ(U)| ≥ |U| = |T|+|S|. So, |Γ(T)∩(Y-Γ(S))| ≥ |T|
- Hence ∃ a complete matching from X into Y ✓

Hall's Theorem

- Claim: No shrinking $S \subseteq X \Rightarrow \exists$ a complete matching from X into Y
- Proof by strong induction on $|X|$.
- Base case, $|X|=1$: ✓ (How?)
- Induction step: Suppose claim holds for graphs with $|X| \leq k$.
  - Given graph (X,Y,E) with $|X|=k+1$, s.t. $\forall U \subseteq X$, $|\Gamma(U)| \geq |U|$
  - Pick an arbitrary $x \in X$, and an arbitrary neighbour y of x (by Hall's condition, {x} is not shrinking, and so has a neighbour).
  - Case 1: There is a complete matching from X−{x} to Y−{y}.
    Then, X has a complete matching into Y ✓
  - Case 2: No complete matching from X−{x} to Y−{y}.
    Then too X has a complete matching into Y. ✓

Example Application

- Claim: The edge set of any bipartite graph in which all the nodes have the same degree d can be partitioned into d matchings
  - Note that such a graph G=(X,Y,E) would have $|X|=|Y|=|E|/d$.
- Proof by induction on d.
- For d=1, the graph is a matching. Suppose holds for $d \leq k$.
- Given a bipartite graph G=(X,Y,E) of degree d=k+1. Enough to find one complete matching M in G.
  - After removing it, will be left with a bipartite graph with degree k for all nodes, and then can use ind. hyp.
- To find one matching, enough to show that no $S \subseteq X$ is shrinking
- $d|S|$ = #edges incident on S ≤ #edges incident on $\Gamma(S) = d|\Gamma(S)|$
  $\Rightarrow |\Gamma(S)| \geq |S|$ ✓

Vertex Cover

- A vertex cover of a graph G=(V,E) is a set C of vertices such that every edge is covered by (incident on) at least one vertex in C
  - i.e., $C \subseteq V$ is a vertex cover if $\forall\ e \in E,\ e \cap C \neq \emptyset$
- Trivial vertex covers: V is a vertex cover. So is $V-\{v\}$, for any $v \in V$
- Algorithmic task: Find a small vertex cover of a given graph
  - "Hard" (i.e., NP-hard) to find the size of smallest vertex cover
- Two useful results connecting the minimum vertex cover problem to the maximum matching problem (which is not a hard problem)
  - In bipartite graphs, the size of a smallest vertex cover equals the size of a largest matching
  - In general graphs, they are within a factor of 2 of each other
  - Next time