Graphs

Lecture 16

Matchings and Vertex Cover

- A matching in a graph G=(V,E) is a set of edges which do not share any vertex
  - i.e., a set $M \subseteq E$ s.t. $\forall e_1,e_2 \in M$, $e_1 \neq e_2 \rightarrow e_1 \cap e_2 = \emptyset$
- **Hall's Theorem: Bipartite graph G=(X,Y,E) has a complete matching from X to Y iff no subset of X is shrinking**
- A vertex cover of a graph G=(V,E) is a set C of vertices such that every edge is covered by (incident on) at least one vertex in C
  - i.e., $C \subseteq V$ is a vertex cover if $\forall\ e \in E$, $e \cap C \neq \emptyset$
- In any graph, $\forall$ vertex cover C, $\forall$ matching M, $|C| \geq |M|$.
  [any vertex can cover at most one edge in M.]
- **Kőnig-Egerváry Theorem: In a bipartite graph, the size of the smallest vertex cover equals the size of the largest matching**

Vertex Cover in Bipartite Graphs

- **Kőnig-Egerváry theorem: In a bipartite graph, the size of the smallest vertex cover equals the size of the largest matching**
- To prove that in a bipartite graph $G=(X,Y,E)$, given a smallest vertex cover C, there is a matching M with $|M| \geq |C|$
- Let $A=C\cap X$ and $B=C\cap Y$. Enough to show $\exists$ a complete matching from A to $Y-B$ and $\exists$ a complete matching from B to $X-A$
  - By Hall's theorem, enough to show that no $S\subseteq A$ is shrinking in $Y-B$ (and similarly that no $S\subseteq B$ is shrinking in $X-A$)
    - Suppose $S\subseteq A$ shrinking in $Y-B$. $C\cup\Gamma(S)-S$ is a vertex cover [edges covered by S are covered by $\Gamma(S)$] and strictly smaller than C! [$|C\cup\Gamma(S)-S| = |C| + |\Gamma(S)-B| - |S| < |C|$.]

Vertex Cover in General Graphs

- Recall that finding (the size of) a smallest Vertex Cover is hard, but finding a maximum matching isn't
  - Even easier to find a <u>maximal</u> matching: M is a maximal matching if no edge $e \in E-M$ such that $M \cup \{e\}$ is also a matching
    - Repeat until no more edges: pick an arbitrary edge, and delete all edges touching it
- If M is a maximal matching, there is a vertex cover of size $2|M|$
  - Include both end points of each edge in M (i.e., $C = \cup_{e \in M} e$)
  - M is maximal $\Rightarrow$ no edge e with both its nodes not in C
    $\Rightarrow$ C is a vertex cover
- If C is a smallest vertex cover and M a maximal matching, $|M| \leq |C| \leq 2|M|$. Hence, can efficiently approximate the size of the smallest vertex cover within a factor of 2.

Question

AQQY

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02:00

- Let $M(G)$ denote the size of the largest matching and $C(G)$ the size of the smallest vertex cover of G. Then

A. $M(K_n) = C(K_n) = \lfloor n/2 \rfloor$
B. $M(K_n) + C(K_n) = n$
C. $M(K_n) = C(K_n)$ iff $n \leq 2$
D. $M(K_n) < C(K_n)$
E. $M(K_n) > C(K_n)$

$M(K_n) = \lfloor n/2 \rfloor$
$C(K_n) = n-1$

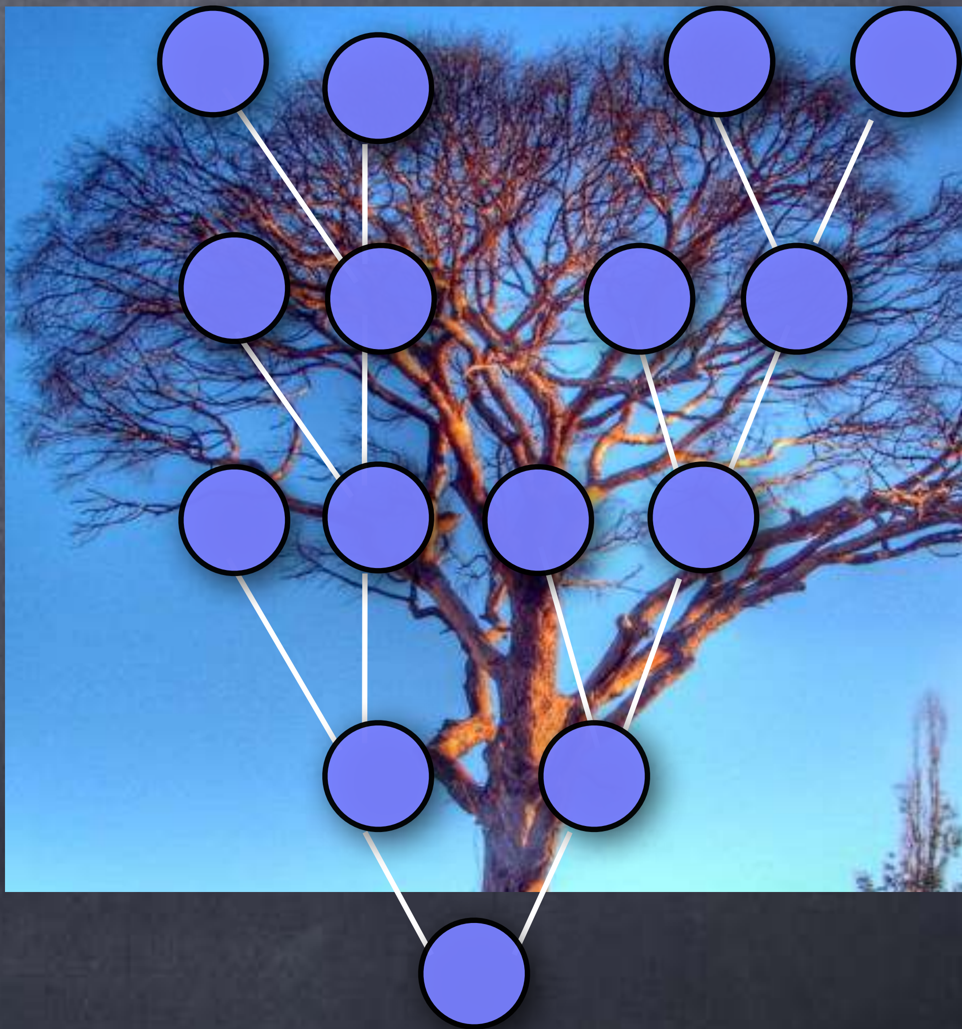
$M(G) = C(G)$ for bipartite G, but K_n for $n>2$ is not bipartite

Vertex Cover and Independent Set

- In a graph $G=(V,E)$, $I \subseteq V$ is said to be an independent set if there are no edges in the subgraph induced by I
  - i.e., $\forall e \in E, \quad e \not\subseteq I$
- I is an independent set iff $V-I$ is a vertex cover
  - $e \not\subseteq I \longleftrightarrow e \cap (V-I) \neq \emptyset$
  - I i.s. $\Leftrightarrow \forall e \in E,\ e \not\subseteq I \Leftrightarrow \forall e \in E,\ e \cap (V-I) \neq \emptyset \Leftrightarrow V-I$ v.c.
- Hence size of smallest i.s. = n - size of largest v.c.

Trees and Forests

- Tree: a connected acyclic graph
- Forest: an acyclic graph
  - Each connected component in a forest is a tree
  - Any subgraph of a tree is a forest (possibly a tree)
    - A single tree is a forest too



Leafs

- A leaf is a node which has degree 1
- Every tree with at least 2 nodes has at least 2 leaves
  - Consider a maximal path $P = v_0,\ldots,v_k$ [exists in any finite graph]
  - $k>0$ [else v_0 is an isolated vertex, and the graph is not connected]
  - If v_0 is not a leaf, it has a neighbour v_i for $i>1$. But then $v_0,\ldots,v_i$ form a cycle! So v_0 is a leaf. Similarly, v_k is a leaf.
- If G is a tree with at least 2 nodes, deleting a leaf w (and the one edge incident on it) results in a tree G′
  - G′ is connected, because all u-v paths in G are retained in G′ for $u,v \neq w$

Induction on Trees (By Deleting Leafs)

- Claim: In a tree, for any two nodes u,v, there is exactly one u-v path (i.e., path from u to v)
- Proof by induction on the number of nodes
- Base case: 1 node. Only one path from v to itself (of length 0) ✓
- Suppose the claim holds for trees with k nodes, for some $k \geq 1$.
- Given a tree G with k+1 nodes, delete a leaf w to get a tree G′
  - (Recall: There is a leaf, and deleting it gives a tree)
  - For $u,v \neq w$: any u-v path in G is present in G′ (w cannot occur in the middle of a path). So, by ind. hyp. exactly one u-v path.
  - For $u \neq w$, $v = w$: Any u-w path in G is of the form u-x path followed by w, where x is w's only neighbour. But exactly one u-x path. So exactly one u-w path.
  - Also, only one w-w path.
  - So for all u,v, exactly one u-v path in G ✓

Number of Edges

- In a tree (V,E), $|E| = |V|-1$
- Proof by induction on $|V|$
- Base case: $|V| = 1$. Only one such tree, and it has $|E|=0$.
- Induction step: for all $k > 1$

  Hypothesis: for every tree (V,E) with $|V|=k-1$, $|E|=|V|-1$

  To prove: for every tree (V,E) with $|V|=k$, $|E|=|V|-1$

  - Suppose $G=(V,E)$ is a tree with $|V| = k > 1$. Consider $G'=(V',E')$ be the tree obtained by deleting a leaf.
  - By induction hypothesis, $|E'|=|V'|-1=k-2$. But $|E|=|E'|+1$ (because exactly one edge was deleted). So $|E|=k-1$.
- In a forest (V,E), the number of connected components, $c=|V|-|E|$
  - Components be (V_i,E_i). Note that $|V| = \Sigma_i |V_i|$ and $|E| = \Sigma_i |E_i|$

    $|E| = \Sigma_{i=1 \text{ to } c} |E_i| = \Sigma_{i=1 \text{ to } c} (|V_i|-1) = (\Sigma_{i=1 \text{ to } c} |V_i|) - c = |V|-c$

ZLGF

Question

02:00

- Suppose |V|=4. How many different trees are there over V?

  A. 4
  B. 8
  C. 12
  D. 16
  E. 20

A tree on V has 3 out of 6 possible edges. C(6,3)=20 such choices. But 4 of them are not trees

Dilworth's Theorem

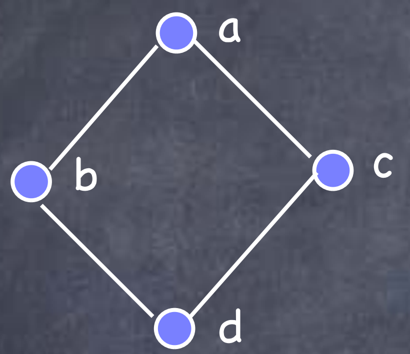
- An application of Kőnig-Egerváry theorem to posets
- **Dilworth's Theorem: In a poset, the size of a largest anti-chain** (called the width of the poset) **equals the size of a smallest chain decomposition** (i.e., a partition of the poset into chains)
- cf. Mirsky's Theorem: Size of a largest chain (height of the poset) equals the size of a smallest anti-chain decomposition
- Easy part: Any chain decomposition is larger than any anti-chain (as no two elements in the anti-chain can be in the same chain)
- Non-trivial part: There is a chain decomposition and an anti-chain of the same size

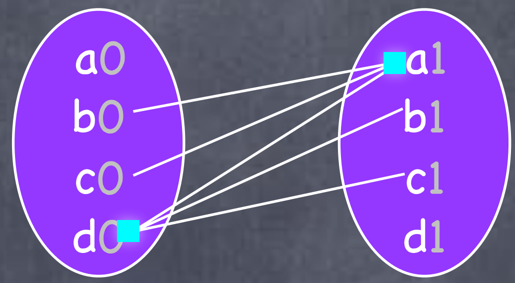
Dilworth's Theorem

- To show that there is an anti-chain at least as large as a chain decomposition (then, must be of the same size)
  - Consider a poset $(S,\preccurlyeq)$, with $|S|=n$
  - Construct a bipartite graph G s.t.
    - a vertex cover of size $\leq t$ in G $\Rightarrow$ antichain of size $\geq n-t$
    - a matching of size $\geq t$ in G $\Rightarrow$ partition S into $\leq n-t$ chains
  - Kőnig-Egerváry theorem: there is a vertex cover and matching of the same size, say t, in G
  - Hence an antichain at least as large as a chain decomposition

Dilworth's Theorem

- Let $G=(S\times\{0\},S\times\{1\},E)$, where $E = \{ \{(u,0),(v,1)\} \mid u\preccurlyeq v,\ u\neq v \}$





$C=\{(d,0),(a,1)\}$
$B = \{a,d\}$
$A = \{b,c\}$

- Given vertex cover C, let $B = \{ u \mid \exists b\in\{0,1\},\ (u,b) \in C \}$. Let $A=S-B$.
  - $|B| \leq |C| \Rightarrow |A| \geq |S|-|C|$
  - Also, A is an anti-chain
    [If $u,v\in A$, and $u\preccurlyeq v$, then $(u,0)$ and $(v,1) \notin C$, and edge $\{(u,0),(v,1)\} \in E$!]

Dilworth's Theorem

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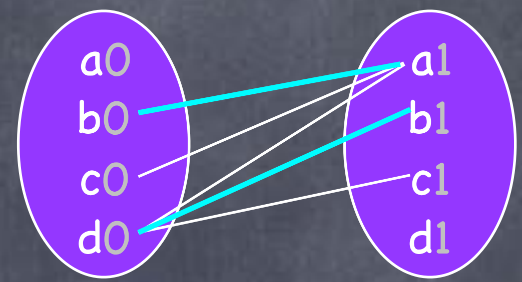

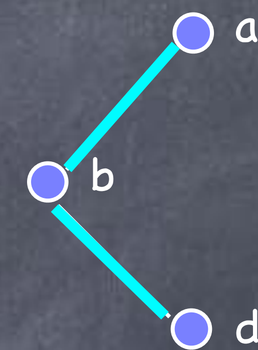

- Given a matching M, define a graph $F=(S,E^*)$, where $E^*=\{ \{u,v\} \mid \{(u,0),(v,1)\} \in M \}$.
  - F is a forest, with each connected component being a path
    - In F, u can have degree at most 2 (one from (u,0) and one from (u,1)). F has no cycles [Cycle $v_0,v_1,\ldots,v_k \Rightarrow v_0 \preccurlyeq v_1 \preccurlyeq \ldots \preccurlyeq v_0$!]
  - Each such path in F forms a chain in the poset
  - Number of chains = number of connected components $= |S| - |E^*| = |S|-|M|$

Dilworth's Theorem

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Min-Max Results

- We saw (easy relations)
  - In a poset, size of any chain ≤ size of any anti-chain decomp.
  - In a poset, size of any anti-chain ≤ size of any chain decomp.
  - In a graph, size of any matching ≤ size of any vertex cover
- Sometimes these turn out to be "tight": Equality can be achieved
  - Mirsky's theorem
  - Dilworth's theorem
  - Kőnig-Egerváry theorem (for bipartite graphs only)