Recursive Definitions

And Applications to Counting

Lecture 17

Tower of Hanoi

- Move entire stack of disks to another peg
  - Move one from the top of one stack to the top of another
  - A disk cannot be placed on top of a smaller disk
- How many moves needed?
- Optimal number not known when 4 pegs and over ≈30 disks!
- Optimal solution known for 3 pegs (and any number of disks)

Tower of Hanoi

- Recursive algorithm (optimal for 3 pegs)
  - Transfer(n,A,C):

    If n=1, move the single disk from peg A to peg C
    Else
        Transfer(n-1,A,B) (leaving the largest disk out of play)
        Move largest disk to peg C
        Transfer(n-1,B,C) (leaving the largest disk out of play)

Tower of Hanoi

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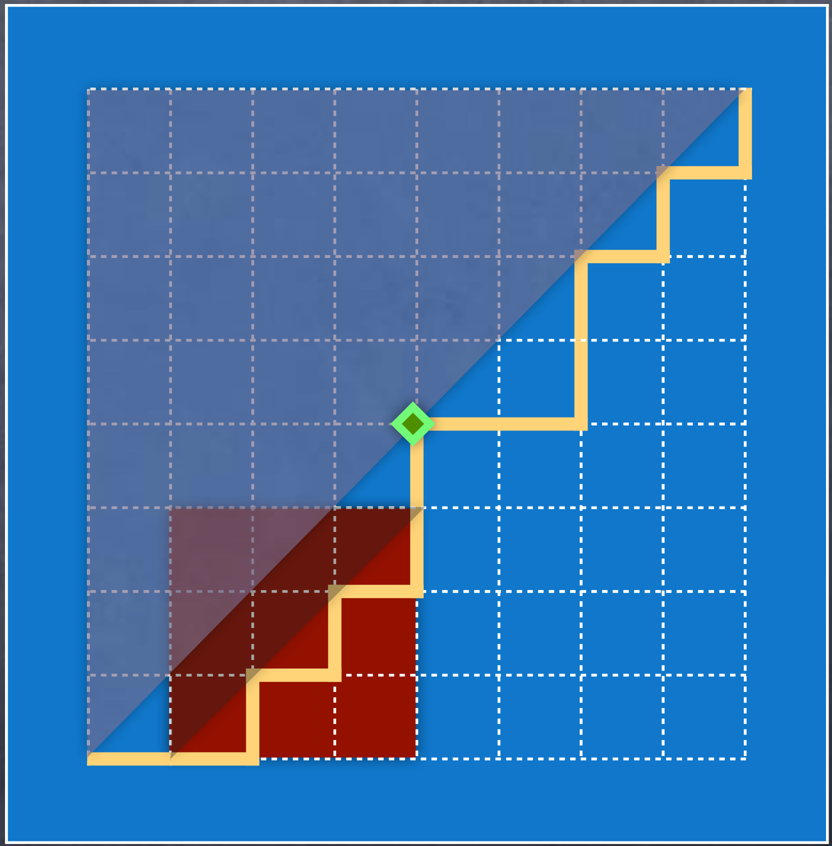
- How many moves are made by this algorithm?
- M(n) be the number of moves made by the above algorithm
- $M(n) = 2M(n-1) + 1$ with $M(1) = 1$
- So, $M(n) = ?$

Catalan Numbers

- How many paths are there in the grid from (0,0) to (n,n) without ever crossing over to the y>x region?



- Any path can be constructed as follows
  - Pick minimum k>0 s.t. (k,k) reached
  - $(0,0) \longrightarrow (1,0) \Rrightarrow (k,k\text{-}1) \longrightarrow (k,k) \Rrightarrow (n,n)$ where $\Rrightarrow$ denotes a Catalan path
- $\mathrm{Cat}(n) = \Sigma_{k=1 \text{ to } n} \mathrm{Cat}(k\text{-}1)\cdot\mathrm{Cat}(n\text{-}k)$
- $\mathrm{Cat}(0) = 1$
- So, $\mathrm{Cat}(n) = ?$

Recursive Definitions

- E.g., $f(0) = 1$ — Initial Condition
  $f(n) = n \cdot f(n-1) \quad \forall n \in \mathbb{Z}$ s.t. $n>0$ — Recurrence relation
- $f(n) = n \cdot (n-1) \cdot \ldots \cdot 1 \cdot 1 = n!$
- This is the formal definition of n! (without using "…")
- A recursive program to compute factorial:

```
factorial(n∈ℕ)  {
   if (n==0) return 1;
   else return n*factorial(n-1);
}
```

YSQR

Question

01:00

- $f(0) = 5$; $f(n) = 3 \cdot f(n-1)$ for $n \in \mathbb{Z}^+$. Then for $n \in \mathbb{N}$

  A. $f(n) = 5^{n+1}$

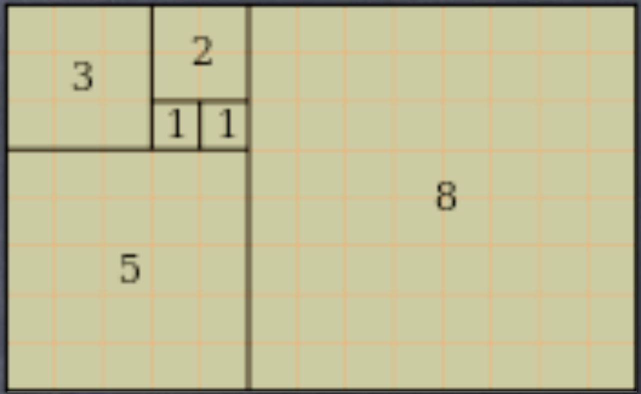
  B. $f(n) = 3 \cdot 5^n$

  C. $f(n) = 5 \cdot 3^n$

  D. $f(n) = 15 \cdot 3^n$

  E. None of the above

Fibonacci Sequence



- $F(0) = 0$
  $F(1) = 1$
  $F(n) = F(n-1) + F(n-2) \quad \forall n \geq 2$
- $F(n)$ called the n^{th} Fibonacci number (starting with 0^{th})

Counting Strings

- How many ternary strings of length n which don't have "00" as a substring?
- Set up a recurrence
  - $A(n)$ = # such strings starting with 0
  - $B(n)$ = # such strings not starting with 0
  - $A(n) = B(n-1)$. $B(n) = 2(A(n-1) + B(n-1))$. [Why?]
- Initial condition: $A(0) = 0$; $B(0) = 1$ (empty string)
- Required count: $A(n) + B(n)$
- Can rewrite in terms of just B
  - $B(0) = 1$. $B(1) = 2$. $B(n) = 2B(n-1) + 2B(n-2) \quad \forall n \geq 2$
  - Required count: $B(n-1) + B(n)$.

JAWS

Question

- Consider bit strings of length n which have "01" as a substring. Let A(n) = # such strings starting with 0 and B(n) = # such strings starting with 1. Then:

A. $A(n) = A(n-1) + B(n-1)$

B. $A(n) = A(n-1) + 2^{n-2}$

C. $A(n) = 2^{n-1} + B(n-1)$

D. $A(n) = 1 + B(n-1)$

E. None of the above

$A(n) = A(n-1) + 2^{n-2}$.
$B(n) = A(n-1) + B(n-1)$

Exercise: Count directly, by counting "bad" strings

Closed Form

- Sometimes possible to get a "closed form" expression for a quantity defined recursively (in terms of simpler operations)
  - e.g., $f(0)=0$ & $f(n) = f(n-1) + n, \forall n>0$
    - $f(n) = n(n+1)/2$
- Sometimes, we just give it a name
  - e.g., n!, Fibonacci(n), Cat(n)
  - In fact, formal definitions of integers, addition, multiplication etc. are recursive
    - e.g., $0 \cdot a = 0$ & $n \cdot a = (n-1) \cdot a + a, \forall n>0$
    - e.g., $2^0 = 1$ & $2^n = 2 \cdot 2^{n-1}$
- Sometimes both
  - e.g., Fibonacci(n), Cat(n) have closed forms (later)

Understanding a Recursive Definition

- Suppose $g(1) = 1$ & $g(n) = 2\, g(n-1) + n \quad \forall n>1$.
  - $g(n)$ is growing "exponentially" by (more than) doubling for each increment in n
  - $g(n) = ?$
- Make a "guess". Then prove by induction
- How do we guess? (More ideas later.)
  - $$\begin{aligned} g(n) &= n + 2\cdot g(n-1) \\ &= n + 2\cdot(\, (n-1) + 2\cdot g(n-2)\,) \\ &= n + 2\cdot(\, (n-1) + 2\cdot(\, (n-2) + 2\cdot g(n-3)\,) \\ &= n + 2.(n-1) + 2^2.(n-2) + 2^3.g(n-3) \end{aligned}$$
  - $g(n) = \Sigma_{k=0 \text{ to } n-1}\ 2^k\cdot(n-k)$ (make sure the base case matches)

Recursion & Induction

- Claim: $F(3n)$ is even, where $F(n)$ is the n^{th} Fibonacci number, $\forall n \geq 0$

0 1 1 **2** 3 5 **8** 13 21 **34**...
Stronger claim (but easier to prove by induction): $F(n)$ is even iff n is a multiple of 3

- Proof by induction:
- Base case:
  $n=0$: $F(3n) = F(0) = 0$ ✔ $n=1$: $F(3n) = F(3) = 2$ ✔
- Induction step: for all $k \geq 2$
  Induction hypothesis: suppose for $0 \leq n \leq k-1$, $F(3n)$ is even
  To prove: $F(3k)$ is even
  - $F(3k) = F(3k-1) + F(3k-2) = ?$
    - Unroll further: $F(3k-1) = F(3k-2) + F(3k-3)$
      $F(3k) = 2 \cdot F(3k-2) + F(3(k-1))$ = even, by induction hypothesis

Recursion & Induction

Example: Fibonacci numbers

- $f(0) = c.\quad f(1) = d.\quad f(n) = a\cdot f(n-1) + b\cdot f(n-2)\quad \forall n\geq 2.$
- Suppose $X^2 - aX - b = 0$ has two distinct (possibly complex) solutions, x and y

Characteristic equation:
replace $f(n)$ by X^n in the recurrence

- Claim: **$f(n) = p\cdot x^n + q\cdot y^n$ for some p,q**
- Base cases satisfied by $p=(d-cy)/(x-y)$, $q=(d-cx)/(y-x)$
- Inductive step: for all $k\geq 2$

  Induction hypothesis: $\forall n$ s.t. $1 \leq n \leq k-1$, $f(n) = px^n + qy^n$

  To prove: $f(k) = px^k - qy^k$

  - $f(k) = a\cdot f(k-1) + b\cdot f(k-2)$

$$= a\cdot(px^{k-1}+qy^{k-1}) + b\cdot(px^{k-2}+qy^{k-2}) - px^k - qy^k + px^k + qy^k$$

$$= - px^{k-2}(x^2-ax-b) - qy^{k-2}(y^2-ay-b) + px^k + qy^k = px^k + qy^k \checkmark$$

Recursion & Induction

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2) \quad \forall n \geq 2$.
- Suppose $X^2 - aX - b = 0$ has only one solution, $x \neq 0$.
  i.e., $a=2x$, $b=-x^2$, so that $X^2 - aX - b = (X-x)^2$.
- Claim: **$f(n) = (p + q \cdot n)x^n$ for some p,q**
- Base cases satisfied by $p = c$, $q = d/x-c$
- Inductive step: for all $k \geq 2$
  Induction hypothesis: $\forall n$ s.t. $1 \leq n \leq k-1$, $f(n) = (p + qn)y^n$
  To prove: $f(k) = (p+qk)x^k$
  - $f(k) = a \cdot f(k-1) + b \cdot f(k-2)$
    $= a\ (p+qk-q)x^{k-1} + b \cdot (p+qk-2q)x^{k-2} - (p+qk)x^k + (p+qk)x^k$
    $= -(p+qk)x^{k-2}(x^2-ax-b) - qx^{k-2}(ax-2b) + (p+qk)x^k = (p+qk)x^k$ ✓

Solving a Recurrence

- Often, once a correct guess is made, easy to prove by induction
- How does one guess?
- Will see a couple of approaches
  - By unrolling the recursion into a chain or a "rooted tree"
  - Using the "method of generating functions" (next time)

Unrolling a recursion

- Often helpful to try "unrolling" the recursion to see what is happening
- e.g., expand into a chain:
  - $T(0) = 0$ & $T(n) = T(n-1) + n^2$ $\forall n \geq 1$
    - $T(n-1) = T(n-2) + (n-1)^2$, $T(n-2) = T(n-3) + (n-2)^2$, ...
    - $T(n) = n^2 + (n-1)^2 + (n-2)^2 + T(n-3)$ $\forall n \geq 3$
    - $T(n) = \Sigma_{k=1 \text{ to } n}\ k^2 + T(0)$ $\forall n \geq 0$

Another example

- $T(1) = 0$
  $T(N) = T(\lfloor N/2 \rfloor) + 1 \quad \forall N \geq 2$
- Let us consider N of the form 2^n (so we can forget the floor)
- $T(N) = 1 + T(N/2)$
  $= 1 + 1 + T(N/4)$
  $= \ldots$
  $= 1 + 1 + \ldots + T(1)$

How many 1's are there?

A slowly growing function

- $T(2^n) = n$
- $T(N) = \log_2 N$ (or simply $\log N$) for N a power of 2
- General N? T monotonically increasing (by strong induction). So,
  $T(2^{\lfloor \log N \rfloor}) \leq T(N) \leq T(2^{\lceil \log N \rceil})$: i.e., $\lfloor \log N \rfloor \leq T(N) \leq \lceil \log N \rceil$
  - In fact, $T(N) = T(2^{\lfloor \log N \rfloor}) = \lfloor \log N \rfloor$ (Exercise)

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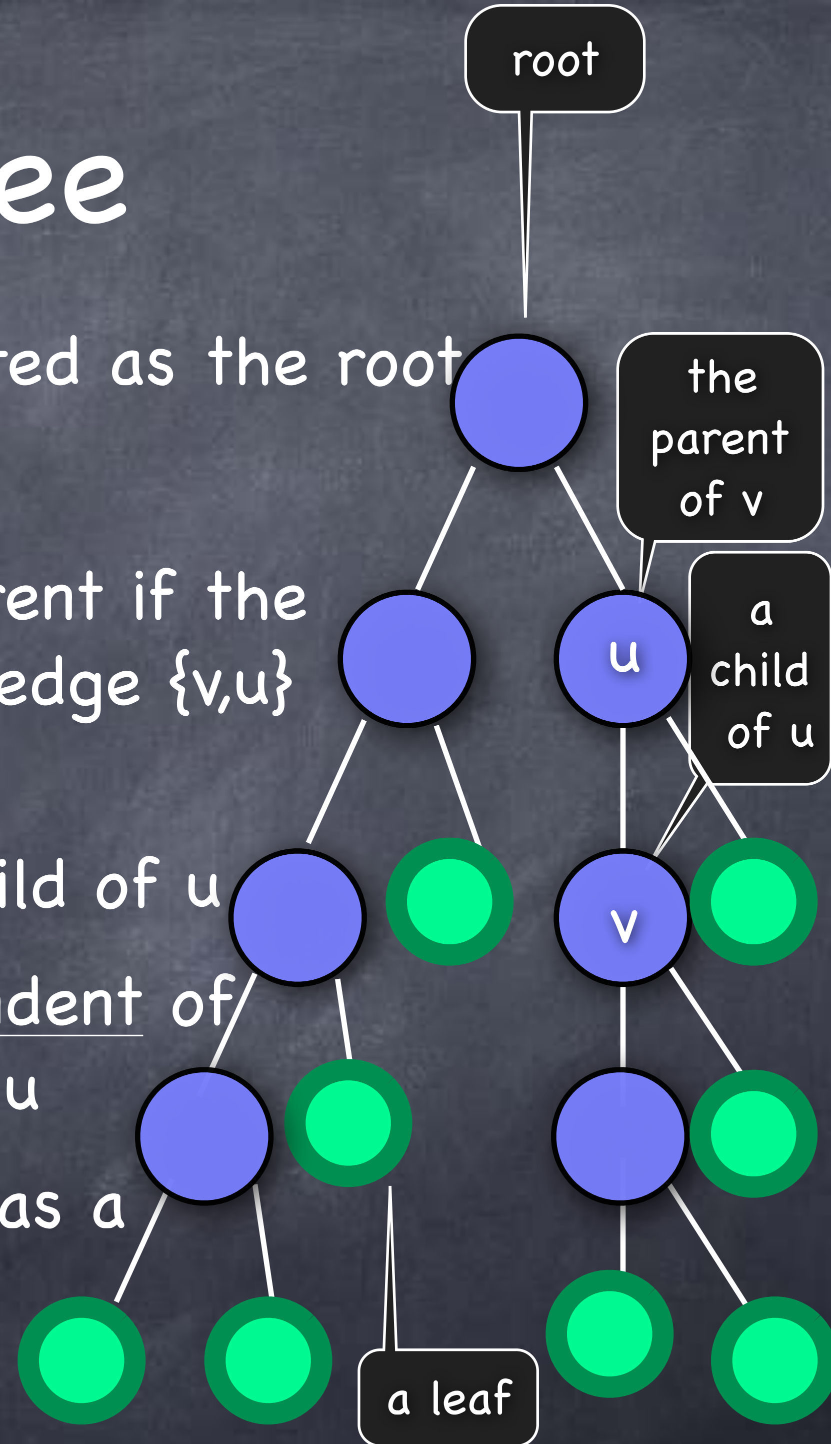
- $M(n)$ be the number of moves made by the above algorithm
- $M(n) = 2M(n-1) + 1$ with $M(1) = 1$
- Unroll the recursion into a "rooted tree"

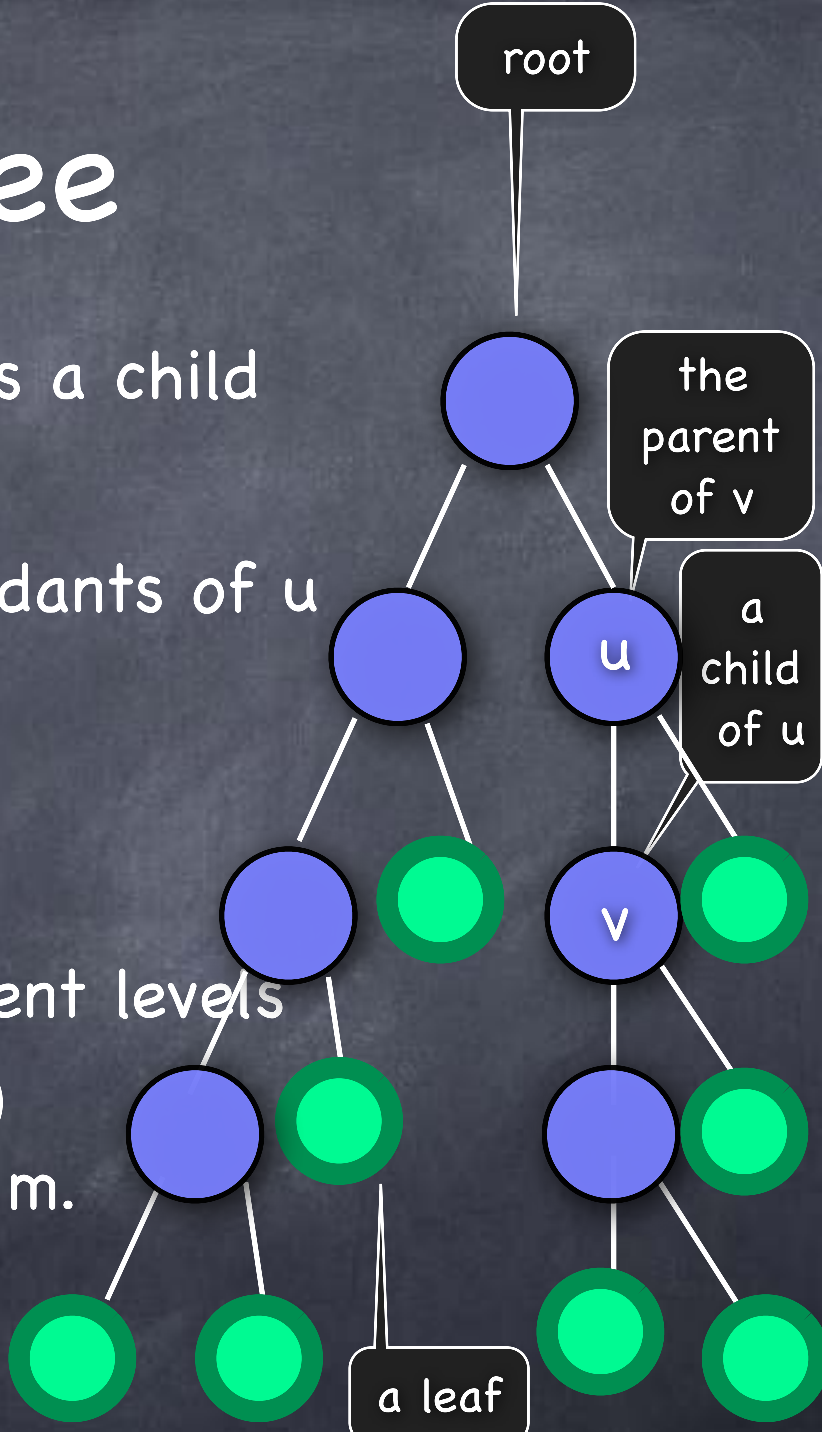
Rooted Tree

- A tree, with a special node, designated as the root
- Typically drawn "upside down"
- Parent and child relation: u is v's parent if the unique path from v to root contains edge {v,u} (parent unique; root has no parent)
  - If u is v's parent v, then v is a child of u
- u is an ancestor of v, and v a descendent of u if the v-root path passes through u
- Leaf is redefined for a rooted tree, as a node with no child
  - Root is a leaf iff it has degree 0 (if deg(root)=1, not called a leaf)



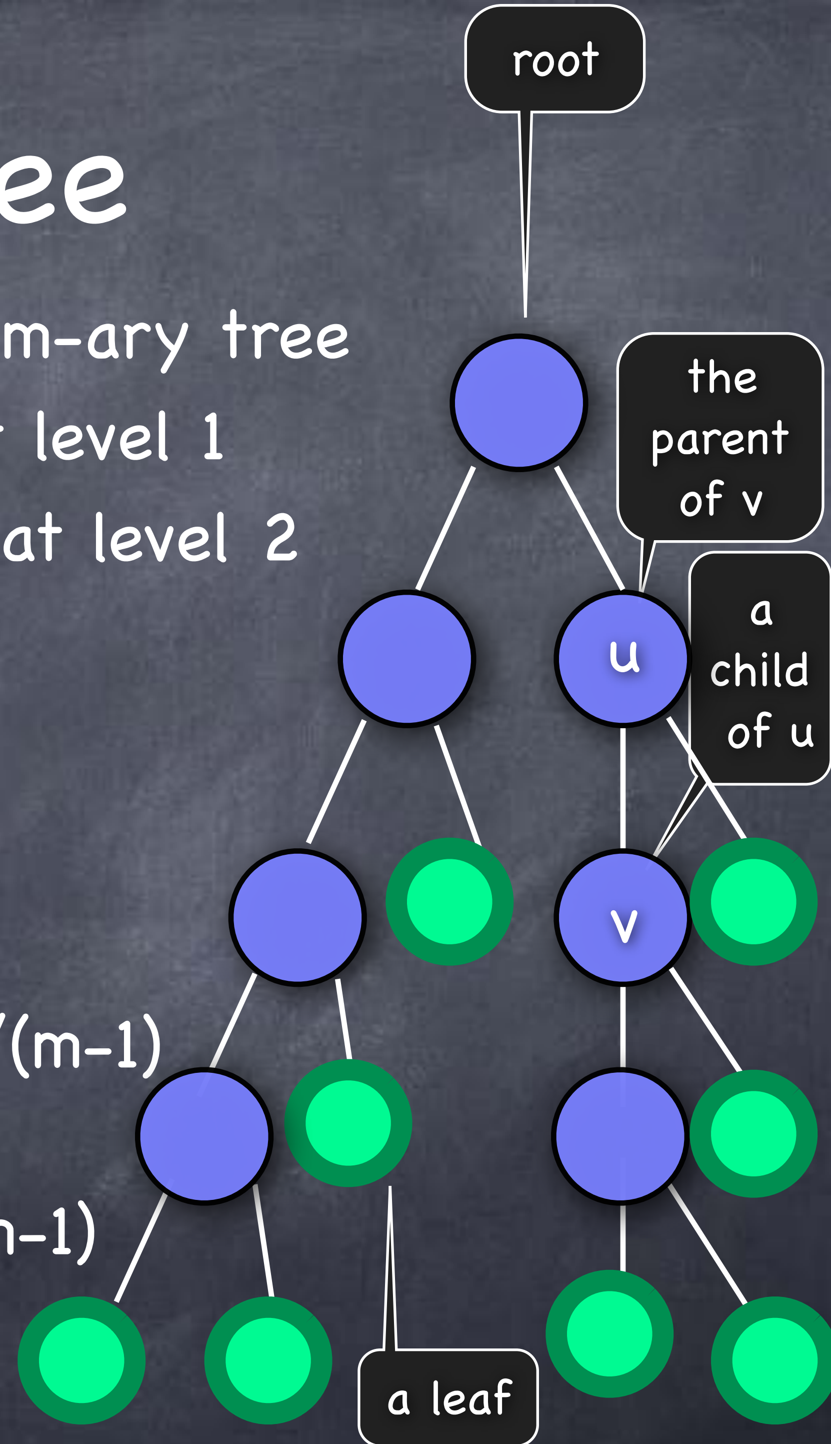
Rooted Tree

- Leaf: no children. Internal node: has a child
- Ancestor, descendant: partial orders
- Subtree rooted at u: with all descendants of u
- Depth of a node: distance from root.
  Height of a tree: maximum depth
- Level i: Set of nodes at depth i.
- Note: tree edges are between adjacent levels
- Arity of a tree: Max (over all nodes) number of children. m-ary if arity $\leq$ m.
- Full m-ary tree: Every internal node has exactly m children.
  Complete & Full: All leaves at same level



Rooted Tree

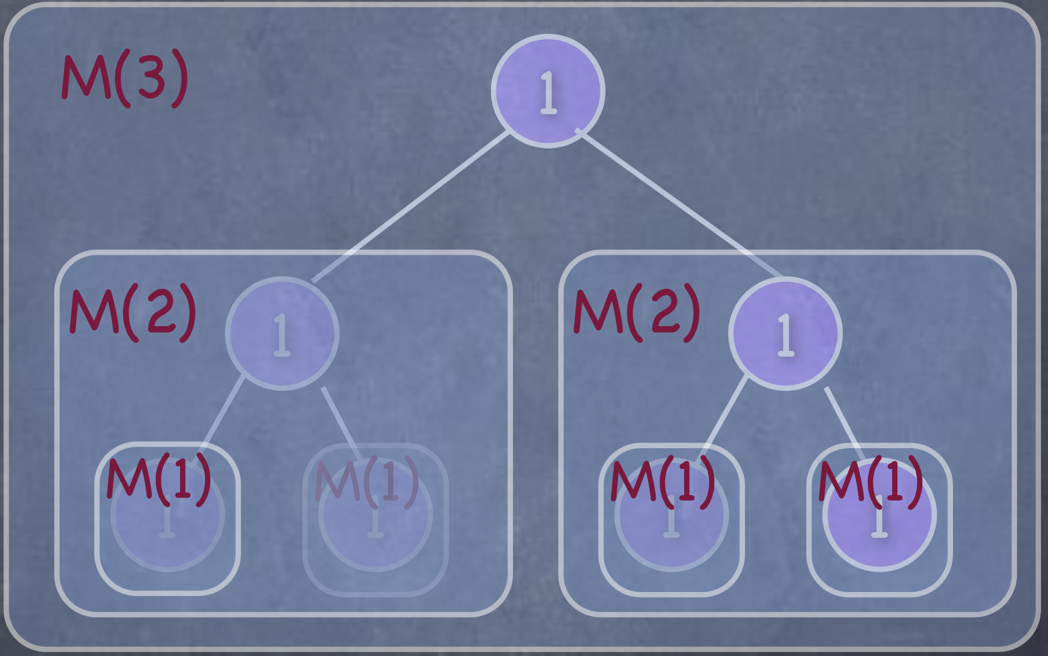
- Number of nodes in Complete & Full m-ary tree
  - One root node with m children at level 1
  - Each level 1 node has m children at level 2
    - m^2 nodes at level 2
  - At level i, m^i nodes
  - m^h leaves, where h is the height
- Total number of nodes:
  - $m^0 + m^1 + m^2 + \dots + m^h = (m^{h+1}-1)/(m-1)$
    - Prove by induction:
      $(m^h-1)/(m-1) + m^h = (m^{h+1}-1)/(m-1)$
- Binary tree (m=2)
  - 2^h leaves, 2^h-1 internal nodes



Tower of Hanoi

- $M(1) = 1$
  $M(n) = 2M(n-1) + 1$

Doing it bottom-up. Could also think top-down



Tower of Hanoi

- $M(1) = 1$
  $M(n) = 2M(n-1) + 1$
- Exponential growth
- $M(2) = 3$, $M(3) = 7$, ...
- $M(n)$ = #nodes in a complete and full binary tree of height $n-1$
- $M(n) = 2^n - 1$