Recursive Definitions

Generating Functions

Lecture 18

Generating Functions

- A generating function is an alternate representation of an infinite sequence, which allows making useful deductions about the sequence (including, possibly, a closed form)
- We will focus on "Ordinary Generating Functions"
- Sequence $f(0), f(1), \ldots$ is represented as the formal expression $G_f(X) \triangleq f(0) + f(1)\cdot X + f(2)\cdot X^2 + \ldots$ (ad infinitum)
  - i.e., for $f : \mathbb{N} \rightarrow \mathbb{R}$, we define $G_f(X) \triangleq \Sigma_{k\geq 0}\, f(k)\cdot X^k$
- e.g., If $f(k) = a^k$ for some $a \in \mathbb{R}$, $G_f(X) = \Sigma_{k\geq 0}\, a^k \cdot X^k$

Generating Functions

- Generating functions sometimes have a succinct representation
- e.g., For $f(k) = a^k$ for some $a \in \mathbb{R}$, $G_f(X) = \sum_{k \geq 0} a^k \cdot X^k$
  - If we substituted for X a real number x sufficiently close to 0, we have $|ax| < 1$ and this would converge to $1/(1-ax)$
  - So we write $G_f(X) = 1/(1-aX)$ (for sufficiently small $|X|$). This will later let us manipulate $G_f(X)$ algebraically

Extended Binomial Theorem

- A useful tool for manipulating/analysing generating functions
- For $a \in \mathbb{R}$, $\binom{a}{k} \triangleq \frac{a(a-1)\ldots(a-k+1)}{k!}$ ($k \in \mathbb{Z}^+$), and $\binom{a}{0} \triangleq 1$
- **Extended binomial theorem:**

  For $|x|<1$, $a \in \mathbb{R}$, $\mathbf{(1+x)^a = \Sigma_{k\geq 0} \binom{a}{k} \cdot x^k}$

- Useful in finding a closed form for f given G_f of certain forms
- e.g., $G_f(X) = 1/(1-X)$. Then, $\Sigma_{k\geq 0}\, f(k) \cdot X^k = (1-X)^{-1}$
  - $\binom{-1}{k} = (-1)(-2)\ldots(-k)/k! = (-1)^k \Rightarrow (1-X)^{-1} = \Sigma_{k\geq 0}\, X^k \Rightarrow f(k)=1$
- Similarly, $\binom{-2}{k} = (-2)(-3)\ldots(-k-1)/k! = (-1)^k(k+1)$

  $\Rightarrow$ $\mathbf{1/(1-X)^2 = \Sigma_{k\geq 0}\, (k+1) \cdot X^k}$

Generating Functions from Recurrence Relations

- Recurrence relations for f often make it easy to compute an expression for the generating function G_f
- e.g., $f(0)=0$, $f(1) = 1$. $f(n) = f(n-1) + f(n-2)$, $\forall n \geq 2$. [Fibonacci]
- $f(n) \cdot X^n = X \cdot f(n-1) \cdot X^{n-1} + X^2 \cdot f(n-2) \cdot X^{n-2}$ (for $n \geq 2$)
  $\Rightarrow G_f(X) = f(0) + f(1) \cdot X + X \cdot (G_f(X) - f(0)) + X^2 \cdot G_f(X)$
  $\Rightarrow G_f(X) (1-X-X^2) = f(0) + (f(1)-f(0)) \cdot X$
  - $G_f(X) = X/(1-X-X^2)$
- More generally:
  $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2)$, $\forall n \geq 2$
  - $G_f(X) = (c + (d-ac)X)/(1-aX-bX^2)$

Generating Functions from Recurrence Relations

- e.g., Let $g(k) = \Sigma_{j=0 \text{ to } k} f(j)$. What is $G_g(X)$, in terms of $G_f(X)$?
  - Recursive definition: $g(0) = f(0)$. $g(n) = g(n-1) + f(n)$, $\forall n \geq 1$.
  - So, $\forall k \geq 1$, $g(k) \cdot X^k = g(k-1) \cdot X^{k-1} \cdot X + f(k) \cdot X^k$
  - $G_g(X) = g(0) + X \cdot G_g(X) + (G_f(X) - f(0))$
  - $G_g(X) = G_f(X)/(1-X)$

Generating Functions for Series Summation

- e.g., $g(k) = \Sigma_{j=0 \text{ to } n} (j+1)^2$
- $G_g(X) = G_f(X)/(1-X)$ where $f(k) = (k+1)^2$
- Consider $G(X) = 1 + X + X^2 + \ldots = 1/(1-X)$
  - $G'(X) = 1 + 2 \cdot X + 3 \cdot X^2 + \ldots = 1/(1-X)^2$
  - Let $H(X) = X\,G(X) = X + 2 \cdot X^2 + 3 \cdot X^3 + \ldots = X/(1-X)^2$
  - So $H'(X) = 1 + 2^2 \cdot X + 3^2 \cdot X^2 + \ldots = 1/(1-X)^2 + 2X/(1-X)^3$
    $= (1+X)/(1-X)^3$
    is the generating function of $f(k) = (k+1)^2$.
- $G_g(X) = (1+X)/(1-X)^4$.
- Now, can use ext. binomial theorem to compute coeff. of X^n

Generating Functions for Counting Combinations

- e.g., Let f(n) = number of ways to throw n balls into d bins (for some fixed number d)
  - Solution 1: Use stars and bars
  - Solution via the generating function $G_f(X)$
    - Coefficient of X^n in $G_f(X)$ must count the number of (non-negative integer) solutions of $n_1 + \ldots + n_d = n$
    - Can write $G_f(X) = (1+X+X^2+\ldots)^d$
    - So, $G_f(X) = [1/(1-X)]^d = (1-X)^{-d}$
    - Coefficient of $X^n = \binom{-d}{n}(-1)^n$

$= d(d+1)\ldots(d+n-1)/n! = C(d+n-1,d-1)$

Generating Functions for Counting Combinations

- e.g., $f(n)$ = #ways to make a total of \$n using \$1, \$5 and \$10 notes. Two variants, f_1 and f_2
  - f_1: order doesn't matter (e.g., $f_1(7)=2$: \$7=2×\$1+1×\$5 and \$7=7×\$1)
  - f_2: order matters (e.g., $f_2(7)=4$ as (5,1,1), (1,5,1), (1,1,5), (1,...,1))
- $G_{f_1}(X) = (1+X+X^2+\ldots)\cdot(1+X^5+X^{10}+\ldots)\cdot(1+X^{10}+X^{20}+\ldots)$
  $= 1/[\ (1-X)\cdot(1-X^5)\cdot(1-X^{10})\ ]$
- $G_{f_2}(X) = ?$
  - Suppose exactly t notes were to be used. #ways to make \$n equals coefficient of X^n in $(X+X^5+X^{10})^t$
  - $G_{f_2}(X) = \Sigma_{t\geq 0}\ (X+X^5+X^{10})^t = 1/(1-(X+X^5+X^{10}))$

Closed Forms

- Goal: find a closed form expression for the coefficient of X^n in G(X), when G(X) has a "nice" expression
- e.g., $G_f(X) = 1/(1-aX) \Rightarrow f(k) = a^k$
- e.g., $G_f(X) = (\alpha + \beta X)/(1-aX-bX^2)$
  - We saw $G_f(X) = (c + (d-ac)X)/(1-aX-bX^2)$
    for: $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2), \forall n \geq 2$
  - Writing $Z = X^{-1}$, we have $G_f(X) = (\alpha Z^2 + \beta Z)/(Z^2-aZ-b)$
  - Let $(Z^2-aZ-b) = (Z-x)(Z-y)$.
  - Two cases: $x \neq y$ and $x=y$

Closed Forms

- $G_f(X) = (\alpha + \beta X)/(1-aX-bX^2) = (\alpha Z^2 + \beta Z)/(Z^2-aZ-b)$, with $Z = X^{-1}$.
- $(Z^2-aZ-b) = (Z-x)(Z-y)$. (So $a=x+y$, $-b=xy$.)
- Case 1: $x \neq y$.
  - $1/(Z^2-aZ-b) = [\ 1/(Z-x) - 1/(Z-y)\ ]/(x-y)$
  - $Z/(Z-x) = 1/(1-xX) = \Sigma_{k\geq 0}\ x^k \cdot X^k$
  - So, $(\alpha Z^2 + \beta Z)/(Z^2-aZ-b) = (\alpha Z + \beta)/(x-y) \cdot \Sigma_{k\geq 0}\ (x^k - y^k) \cdot X^k$
    $= \Sigma_{k\geq 0}\ (\alpha(x^{k+1}-y^{k+1}) + \beta(x^k-y^k))/(x-y) \cdot X^k$
    $= \Sigma_{k\geq 0}\ (px^k+qy^k) \cdot X^k$, where $p=(\alpha x+\beta)/(x-y)$, $q=(\alpha y+\beta)/(y-x)$
  - $f(n)$ = coefficient of $X^n = px^n + qy^n$
- $\alpha=c$, $\beta=d-ac=d-(x+y)c \Rightarrow\ p=(d-yc)/(x-y)$, $q=(d-xc)/(y-x)$,

Recall

Closed Forms

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2)$ $\forall n \geq 2$.
- Suppose $X^2 - aX - b = 0$ has two distinct (possibly complex) solutions, x and y
- Claim: **$f(n) = p \cdot x^n + q \cdot y^n$ for some p,q**
- Base cases satisfied by $p=(d-cy)/(x-y)$, $q=(d-cx)/(y-x)$
- Inductive step: for all $k \geq 2$
  Induction hypothesis: $\forall n$ s.t. $1 \leq n \leq k-1$, $f(n) = px^n + qy^n$
  To prove: $f(k) = px^k - qy^k$
  - $f(k) = a \cdot f(k-1) + b \cdot f(k-2)$
    $= a \cdot (px^{k-1}+qy^{k-1}) + b \cdot (px^{k-2}+qy^{k-2}) - px^k - qy^k + px^k + qy^k$
    $= - px^{k-2}(x^2-ax-b) - qy^{k-2}(y^2-ay-b) + px^k + qy^k = px^k + qy^k$ ✓

Closed Forms

- $G_f(X) = (\alpha + \beta X)/(1-aX-bX^2) = (\alpha Z^2 + \beta Z)/(Z^2-aZ-b)$, with $Z = X^{-1}$.
- $(Z^2-aZ-b) = (Z-x)(Z-y)$. (So $a=x+y$, $-b=xy$.)
- Case 2: $x=y$.
  - $(\alpha Z^2 + \beta Z)/(Z^2-aZ-b) = (\alpha Z^2 + \beta Z)/(Z-x)^2 = (\alpha + \beta X)/(1-xX)^2$
  - Will use $1/(1-aX)^2 = \Sigma_{k\geq 0} (k+1).a^k \cdot X^k$
    - From the extended binomial theorem with $\binom{-2}{k} = (-1)^k(k+1)$
    - Or, by taking derivative of $G(X) = 1/(1-aX) = \Sigma_{k\geq 0}\, a^k \cdot X^k$ we get $G'(X) = a/(1-aX)^2 = \Sigma_{k\geq 1}\, k.a^k \cdot X^{k-1}$
  - $(\alpha + \beta X)/(1-xX)^2 = \Sigma_{k\geq 0} (\alpha + \beta X) \cdot (k+1) \cdot x^k \cdot X^k$

$$= \Sigma_{k\geq 0} (\alpha \cdot (k+1) \cdot x^k + \beta \cdot k \cdot x^{k-1}) \cdot X^k$$

$$= \Sigma_{k\geq 0} (p + qk) x^k \cdot X^k, \text{ where } p=\alpha,\ q=(\alpha+\beta/x)$$

Recall

Closed Forms

- $f(0) = c.\quad f(1) = d.\quad f(n) = a\cdot f(n-1) + b\cdot f(n-2) \quad \forall n \geq 2.$
- Suppose $X^2 - aX - b = 0$ has only one solution, $x \neq 0$. i.e., $a=2x$, $b=-x^2$, so that $X^2 - aX - b = (X-x)^2$.
- Claim: **$f(n) = (p + q\cdot n)x^n$ for some p,q**
- Base cases satisfied by $p = c$, $q = d/x-c$
- Inductive step: for all $k \geq 2$

  Induction hypothesis: $\forall n$ s.t. $1 \leq n \leq k-1$, $f(n) = (p + qn)y^n$

  To prove: $f(k) = (p+qk)x^k$

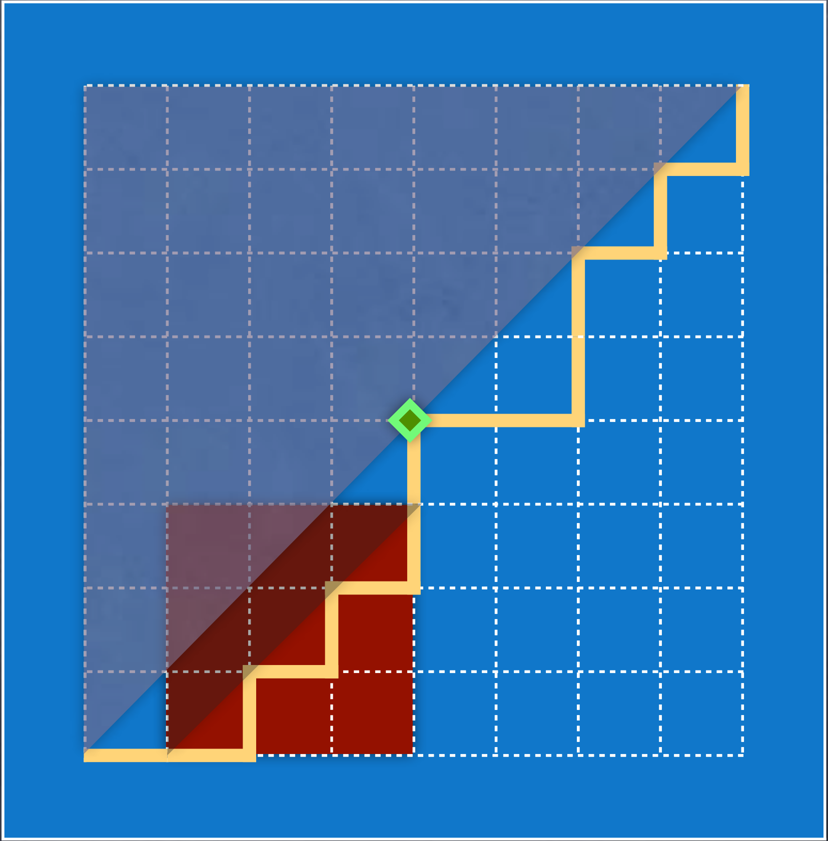
  - $f(k) = a\cdot f(k-1) + b\cdot f(k-2)$

    $= a\,(p+qk-q)x^{k-1} + b\cdot(p+qk-2q)x^{k-2} - (p+qk)x^k + (p+qk)x^k$

    $= -(p+qk)x^{k-2}(x^2-ax-b) - qx^{k-2}(ax-2b) + (p+qk)x^k = (p+qk)x^k$ ✓

Catalan Numbers

- How many paths are there in the grid from (0,0) to (n,n) without ever crossing over to the y>x region?
- Any path can be constructed as follows
  - Pick minimum k>0 s.t. (k,k) reached
  - $(0,0) \longrightarrow (1,0) \Rrightarrow (k,k-1) \longrightarrow (k,k) \Rrightarrow (n,n)$ where $\Rrightarrow$ denotes a Catalan path
- $Cat(n) = \Sigma_{k=1 \text{ to } n}\ Cat(k-1) \cdot Cat(n-k)$
- $Cat(0) = 1$
- So, $Cat(n) = ?$



Catalan Numbers

- $Cat(n) = \Sigma_{k=1 \text{ to } n} Cat(k-1) \cdot Cat(n-k) \quad \forall n \geq =1$
- $Cat(n)\ X^n = \Sigma_{k=1 \text{ to } n} Cat(k-1) \cdot Cat(n-k) \cdot X^n$,

  $= \text{term of } X^n \text{ in } X \cdot (\Sigma_k Cat(k-1)\ X^{k-1}) \cdot (\Sigma_k Cat(n-k)\ X^{n-k}), \quad \forall n \geq =1$

  - For n=0, we have $Cat(0)\ X^0 = 1$
- $G_{Cat}(X) = 1 + X\ G_{Cat}(X)\ G_{Cat}(X)$
- Solving for G in $X \cdot G^2 - G + 1 = 0$, we have $G=[1 \pm \sqrt{(1-4X)}]/(2X)$
  - We need $\lim_{X \to 0} G_{cat}(X) = Cat(0) = 1$

    L'Hôpital's Rule

    - $\lim_{X \to 0} [1 \pm \sqrt{(1-4X)}]/(2X) = \lim_{X \to 0} \pm (-4/[2\sqrt{(1-4X)}])/2 = \pm(-1)$
  - So we take $G_{cat}(X) = [1-\sqrt{(1-4X)}]/(2X)$
  - Then, what is the coefficient of X^n in $G_{cat}(X)$?

Catalan Numbers

- $G_{cat}(X) = [1-\sqrt{(1-4X)}]/(2X)$
- Then, what is the coefficient of X^k in $G_{cat}(X)$?
- Use extended binomial theorem:

$$(1-4X)^{½} = \Sigma_{k\geq 0} \binom{1/2}{k}(-4X)^k = 1 + \Sigma_{k\geq 1} -\binom{2k-2}{k-1} \cdot 2 / k$$

- where $\binom{1/2}{k} = (1/2)(-1/2)(-3/2)(-5/2) \ldots (-(2k-3)/2) /k!$

$$= (-1)^{k-1}(1 \cdot 1 \cdot 3 \cdot \ldots \cdot (2k-3))/[k!\ 2^k] = (-1)^{k-1}\binom{2k-2}{k-1}/[k\ 2^{2k-1}]$$

- $\text{Cat}(k) = \text{Coefficient of } X^k = \binom{2k}{k} \cdot 2/(k+1) \cdot 1/2 = \binom{2k}{k}/(k+1)$