Design & Analysis of Algorithms

The Big O

Lecture 19

How it scales

- In analysing running time (or memory/power consumption) of an algorithm, we are interested in how it scales as the problem instance grows in "size"
  - Running time on small instances of a problem are often not a serious concern (anyway small)
- Also, exact time/number of steps is less interesting
  - Can differ in different platforms. Not a property of the algorithm alone.
  - Thus "unit of time" (constant factors) typically ignored when analysing the algorithm.

How it scales

- So, interested in how a <u>function</u> scales with its input: behaviour on large values, up to constant factors
- e.g., suppose number of "steps" taken by an algorithm to sort a list of n elements varies between 3n and $3n^2+9$ (depending on what the list looks like)
  - If n is doubled, time taken in the worst case could become (roughly) 4 times. If n is tripled, it could become (roughly, in the worst case) 9 times
  - An upper bound that grows "like" n^2

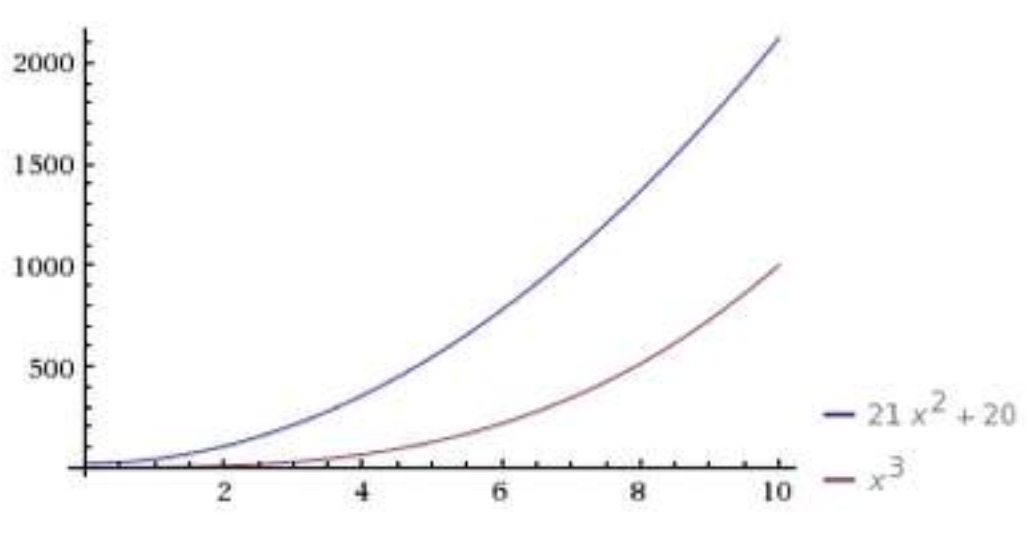
Upper-bounds: Big O

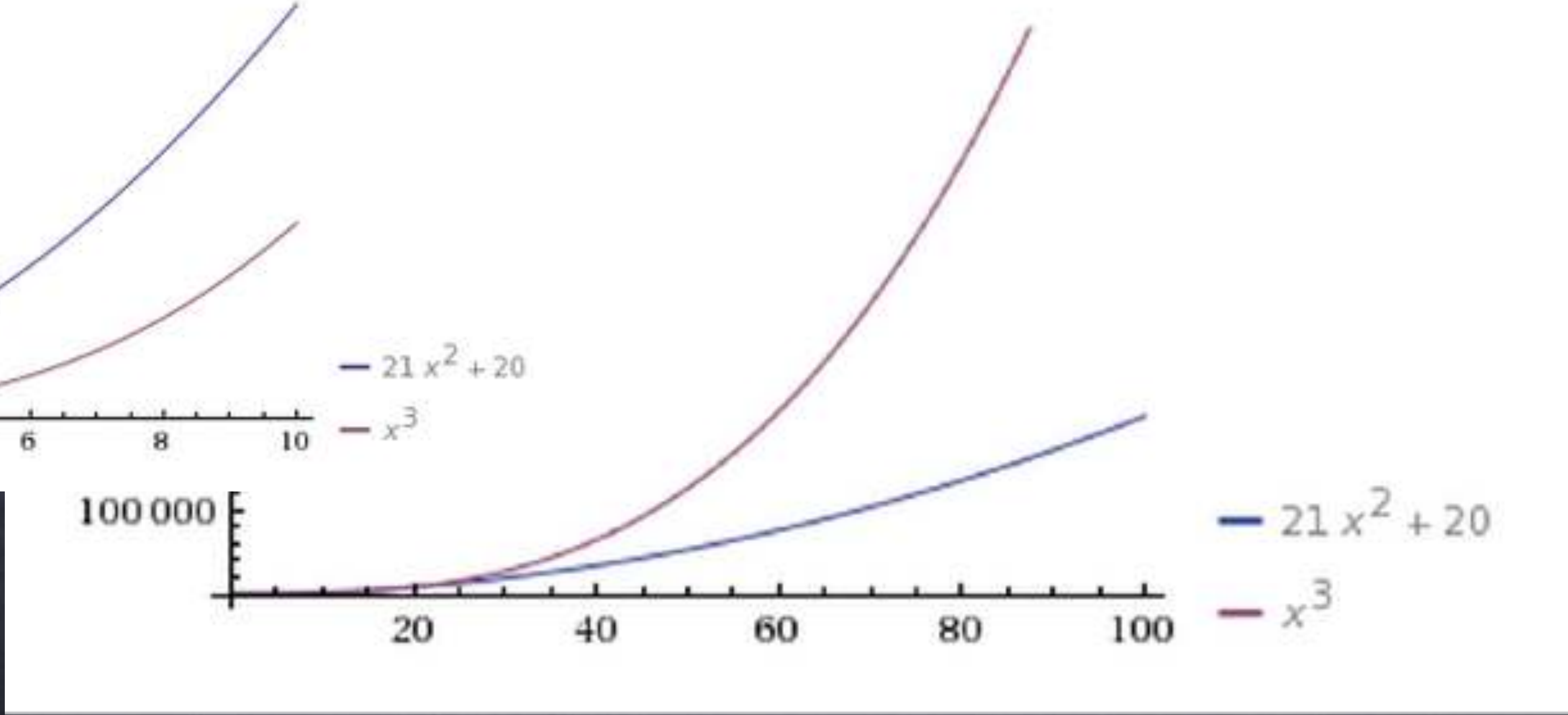
- T(n) has an upper-bound that grows "like" f(n)
  - $T(n) = O(f(n))$
    - $\exists c, k > 0, \quad \forall n \geq k, \quad 0 \leq T(n) \leq c \cdot f(n)$
    - Note: we are defining it only for T & f which are eventually non-negative
    - Note: order of quantifiers! c can't depend on n (that is why c is called a constant factor)
- Important: If $T(n)=O(f(n))$, f(n) could be much larger than T(n) (but only a constant factor smaller than T(n))

Unfortunate notation! An alternative used sometimes: $T(n) \in O(f(n))$

Big-O

- e.g. $T(x) = 21x^2 + 20$
- $T(x) = O(x^3)$





Big-O

- e.g. $T(x) = 21x^2 + 20$
- $T(x) = O(x^3)$
- $T(x) = O(x^2)$ too, since we allow scaling by constants
- But $T(x) \neq O(x)$.
  - $\forall c>0, \forall k>0, \exists x^* \geq k \;\; T(x^*) > c.x^*$

Big-O

- Used in the analysis of running time of algorithms: Worst-case Time(input size) = O(f(input size))
  - e.g. $T(n) = O(n^2)$
- Also used to bound approximation errors
  - e.g., $|\log(n!) - \log(n^n)| = O(n)$
    - A better approximation: $|\log(n!) - \log((n/e)^n)| = O(\log n)$
    - Even better: $|\log(n!) - \log((n/e)^n) - ½\cdot\log(n)| = O(1)$
- We may also have $T(n) = O(f(n))$, where f is a decreasing function (especially when bounding errors)
  - e.g. $T(n) = O(1/n)$

Big O examples

- Suppose $T(n) = O(f(n))$ and $R(n) = O(f(n))$
  - i.e., $\forall n \geq k_T,\ 0 \leq T(n) \leq c_T \cdot f(n)$ and $\forall n \geq k_R,\ 0 \leq R(n) \leq c_R \cdot f(n)$
  - $T(n) + R(n) = O(f(n))$
    - Then, $\forall n \geq \max(k_T, k_R),\ \ 0 \leq T(n)+R(n) \leq (c_R+c_T) \cdot f(n)$
  - If eventually ($\forall n \geq k$), $R(n) \geq 0$, then $T(n) - R(n) = O(T(n))$
    - $\forall n \geq \max(k, k_R),\ \ T(n)-R(n) \leq 1 \cdot T(n)$
- If $T(n) = O(f(n))$ and $f(n) = O(g(n))$, then $T(n) = O(g(n))$
    - $\forall n \geq \max(k_T, k_f),\ 0 \leq T(n) \leq c_T \cdot f(n) \leq c_T c_f \cdot g(n)$
- e.g., $7n^2 + 14n + 2 = O(n^2)$ because $7n^2$, $14n$, 2 are all $O(n^2)$
- More generally, if $T(n)$ is upper-bounded by a degree d polynomial with a positive coefficient for n^d, then $T(n) = O(n^d)$

Some important functions

- $T(n) = O(1)$: $\exists c$ s.t. $T(n) \le c$ for all ~~sufficiently large~~ n
- $T(n) = O(\log n)$. $T(n)$ grows quite slowly, because log n grows quite slowly (when n doubles, log n grows by 1)
- $T(n) = O(n)$: $T(n)$ is (at most) linear in n
- $T(n) = O(n^2)$: $T(n)$ is (at most) quadratic in n
- $T(n) = O(n^d)$ for some fixed d: $T(n)$ is (at most) polynomial in n
- $T(n) = O(2^{d \cdot n})$ for some fixed d: $T(n)$ is (at most) exponential in n. $T(n)$ could grow very quickly.

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STVB

Question

02:00

- Below n denotes the number of nodes in a complete and full 3-ary rooted tree and h its height. Which of the following is/are true, when considering h as a function of n, and n as a function of h?

1. $h = O(\log_3 n)$
2. $h = O(\log_2 n)$
3. $n = O(3^h)$
4. $n = O(2^h)$

A. 1 & 3 only
B. 2 & 4 only
C. 1, 3 & 4 only
D. 1, 2 & 3 only
E. 1, 2, 3 & 4

Theta Notation

- If we can give a "tight" upper and lower-bound we use the Theta notation
  - $T(n) = \Theta(f(n))$ if $T(n)=O(f(n))$ and $f(n)=O(T(n))$
  - e.g., $3n^2-n = \Theta(n^2)$
- If $T(n) = \Theta(f(n))$ and $R(n) = \Theta(f(n))$, $T(n) + R(n) = \Theta(f(n))$

ESBF

Question

- Which of the following is/are true?

1. If $f(x) = O(g(x))$ and $g(x) = O(h(x))$ then $f(x) = O(h(x))$
2. If $f(x) = O(g(x))$ and $h(x) = O(g(x))$ then $f(x) = O(h(x))$
3. If $f(x) = \Theta(g(x))$ and $h(x) = \Theta(g(x))$ then $f(x) = \Theta(h(x))$

   A. 1 only

   B. 1 & 2 only

   C. 3 only

   D. 1 & 3 only

   E. 1, 2 & 3

$\simeq$ and $\ll$

- Asymptotically equal: $f(n) \simeq g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$
  - i.e., eventually, f(n) and g(n) are equal (up to lower order terms)
  - If $\exists c>0$ s.t. $f(n) \simeq c\cdot g(n)$ then $f(n) = \Theta(g(n))$
    (for f(n) and g(n) which are eventually positive)
- Asymptotically much smaller: $f(n) \ll g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$
  - If $f(n) \ll g(n)$ then $f(n) = O(g(n))$ but $f(n) \neq \Theta(g(n))$
    (for f(n) and g(n) which are eventually positive)
- Note: Not necessary conditions: Θ and O do not require the limit to exist (e.g., $f(n) = n$ for odd n and 2n for even n: then $f(n) = \Theta(n)$)

Analysing Algorithms

- Analyse correctness and running time (or other resources)
  - Latter can be quite complicated
- Behaviour depends on the particular inputs, but we often restrict the analysis to worst-case over all possible inputs of the same "size"
  - Size of a problem is defined in some natural way (e.g., number of elements in a list to be sorted, number of nodes in a graph to be coloured, etc.)
  - Generically, could define as number of bits needed to write down the input

Loops

- If an algorithm is "straight-line" without loops or recursion, its running time would be O(1)
- Need to analyse how many times a loop is taken
- e.g. find max among n numbers in an array L

```
findmax(L,n) {
   max = L[1]
   for i = 2 to n {
      if (L[i] > max)
         max = L[i]
   }
   return max
}
```

Time taken by findmax(L,n)
$T(n) = O(n)$

Nested Loops

- If an outer-loop is executed p times, and each time an inner-loop is executed q times, the code inside the inner-loop is executed $p \cdot q$ times in all

- More generally, the number of times the inner-loop is taken can be different in different executions of the outer-loop

- e.g.

```
for i = 1 to n {
    for j = 1 to i {
        tap-fingers()
    }
}
```

what all values of (i,j) are possible when we get here?

i=1: j=1. i=2: j=1,2. i=3: j=1,2,3. ... i=n: j=1,2,..,n.

$1 + 2 + 3 + \ldots + n = n(n+1)/2 = O(n^2)$

Loops

- ```
  i = 1
  while i ≤ n {
      for j = 1 to n {
          tap-fingers()
      }
      i = 2*i
  }
  ```

  $i=1, 2, 4, \ldots, 2^{\lfloor \log n \rfloor}$ ($j=1,2,..,n$ always)

  $O(n \log n)$

- ```
  i = 1
  while i ≤ n {
      for j = 1 to i {
          tap-fingers()
      }
      i = 2*i
  }
  ```

  $i=1, 2, 4, \ldots, 2^{\lfloor \log n \rfloor}$ but $j=1,\ldots,i$

  $1 + 2 + 4 + \ldots + 2^{\lfloor \log n \rfloor} = O(n)$

  Number of nodes in a complete & full binary rooted tree with (about) n leaves