Design & Analysis of Algorithms

The Big O

Lecture 20

Upper-bounds: Big O

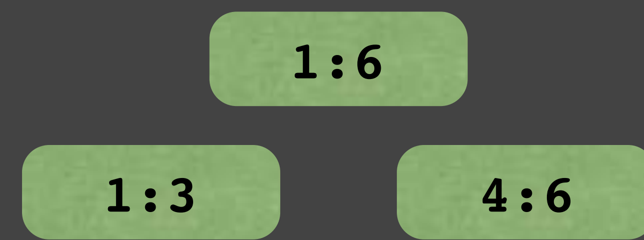
- T(n) has an upper-bound that grows "like" f(n)
  - $T(n) = O(f(n))$
    - $\exists c, k > 0, \quad \forall n \geq k, \quad 0 \leq T(n) \leq c \cdot f(n)$
  - $T(n) = \Theta(f(n))$ if $T(n)=O(f(n))$ and $f(n)=O(T(n))$

Recursion

- Given an array **L**, find max among numbers between positions **start** and **end** (inclusive)

```
findmax (L, start, end) {
    if (start == end)
        return  L[start]
    else {
        mid = ⌊(start+end)/2⌋
        x = findmax(L,start,mid)
        y = findmax(L,mid+1,end)
        if (x>y) return x
        else return y
    }
}
```

e.g. `findmax(L,1,6)`



Correctness by strong induction:
Induct on the size of the problem.
i.e., the length of the list,
$n = |end-start+1|$

How about the running time?

Recursion

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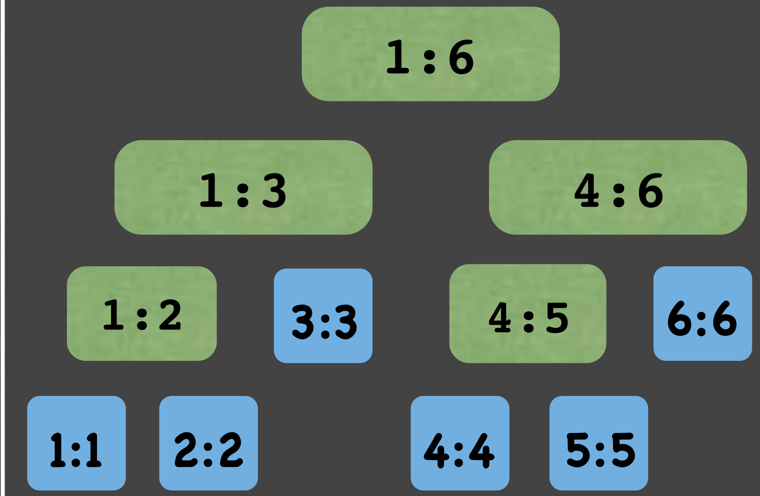

Recursion structure:

A full binary rooted tree with n leaves

(Not important that the split was into almost equal parts)

Recursion

- Given an array **L**, find max among numbers between positions **start** and **end** (inclusive)

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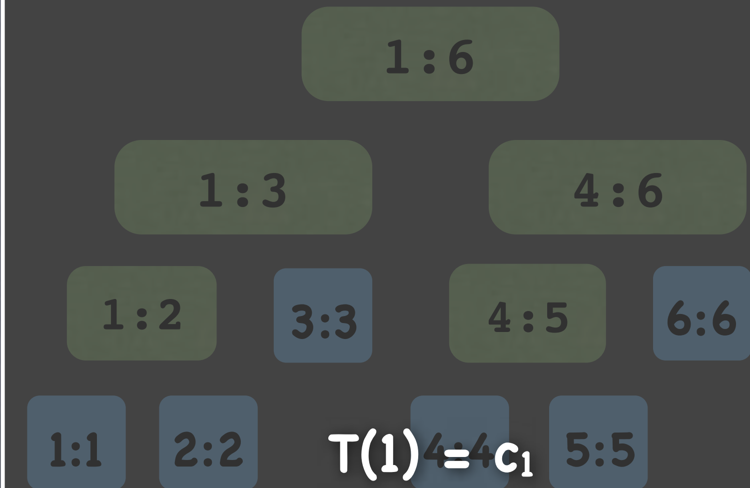
    if (start == end)
        return  L[start]
    else {
        mid = ⌊(start+end)/2⌋
        x = findmax(L,start,mid)
        y = findmax(L,mid+1,end)
        if (x>y) return x
        else return y
    }
}
```

Time T(n) taken by findmax(L,a,a+n-1)?



$T(1) = c_1$

$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_2$

Recursion tree: c_1 on each leaf and c_2 on each internal node

T(n) = O(number of nodes)

T(n) = O(n)

Question

1

VCVP

02:00

- Time taken by find3max(L,a,a+n) is

  A. $\Theta(n)$
  B. $\Theta(n \log n)$
  C. $\Theta(n^{3/2})$
  D. $\Theta(n^3)$
  E. None of the above

```
find3max (L, st, en) {
    if (st == en)
        return  L[st]
    else {
        mid1 = st +⌊(en-st+1)/3⌋
        mid2 = st + 2*⌊(en-st+1)/3⌋
        x = find3max(L,st,mid1)
        y = find3max(L,mid1+1,mid2)
        z = find3max(L,mid2+1,en)
        if (x≥y ∧ x≥z) return x
        if (y≥x ∧ y≥z) return y
        if (z≥x ∧ z≥y) return z
    }
```

$T(n) = \Theta(\#\text{nodes in a full ternary rooted tree with } n \text{ leaves}) = \Theta(n)$

Merge Sort

- Sorting by divide-and-conquer
  - Split the list into two (unless a single element)
  - Sort each list recursively
  - Merge the sorted lists into a single sorted list
- $T(n) = 2T(n/2) + \text{time to merge}$

Merging Two Sorted Lists

- Maintain the invariant that a list K has a prefix of the final merged list. X_1, X_2 have the rest of L_1, L_2.
  - Base case: K=empty, $X_1=L_1$, $X_2=L_2$
  - Inductively, move the smaller of first(X_1) and first(X_2) to the end of K
  - Terminating condition: Both X_1 and X_2 are empty
- Time taken (as a function of $n = |L_1|+|L_2|$) ?
  - When finished K has n elements
  - Each element gets added to K exactly once
  - Each iteration adds exactly one element to K (in O(1) time)
  - T(n) = O(n)

```
merge (L₁, L₂ : ascending lists) {
    K = empty-list; X₁ = L₁; X₂ = L₂;
    while (X₁ not empty or X₂ not empty) {
        if (X₂ empty)
            x = pop(X₁)
        else if (X₁ empty)
            x = pop(X₂)
        else if ( first(X₁) ≤ first(X₂) )
            x = pop(X₁)
        else
            x = pop(X₂)
        append(K,x)
    }
    return K
}
```

Merge Sort

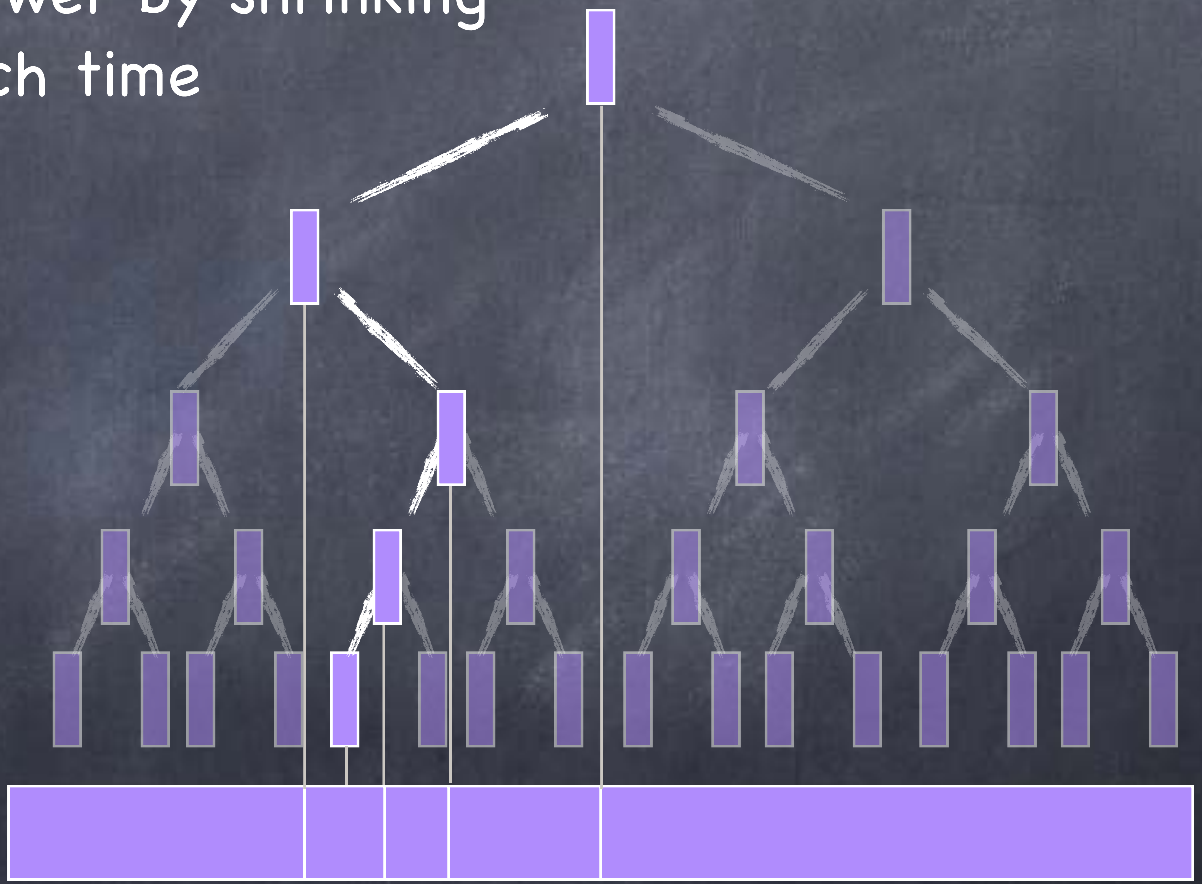
- Sorting by divide-and-conquer
  - Split the list into two (unless a single element)
  - Sort each list recursively
  - Merge the sorted lists into a single sorted list
- $T(n) = 2T(n/2)$ + time to merge
- $T(n) = 2T(n/2) + c\,n$
  - Contribution from each level : $O(n)$
  - Depth of recursion = $O(\log n)$
  - $T(n) = O(n \log n)$

Binary Search

- Find where a desired object occurs (if at all) in a sorted list of objects
  - Objects can be compared with each other (using a total ordering)
- Simple idea:
  - Check if desired object = middle one in the list
  - If not, comparing with the middle one lets you see if it could be in the left half or the right half of the list (since the list is sorted)
  - Recursively search in that half
  - Depth of recursion, for an n element list $\leq \lceil \log_2 n \rceil$

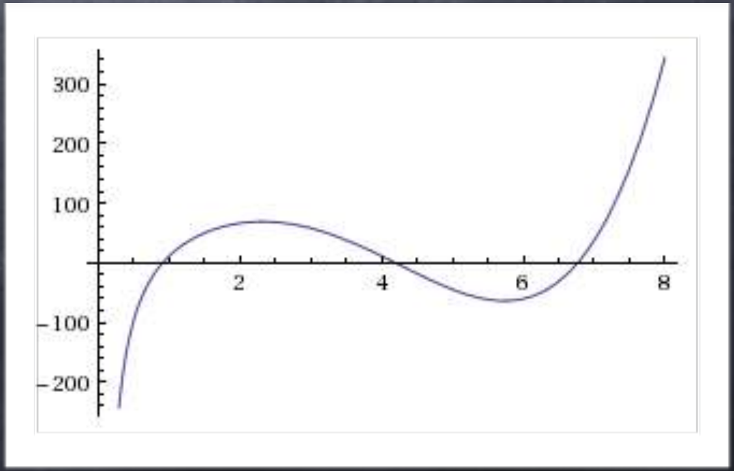
Binary Search

- Zeroing in on the answer by shrinking the range by half each time
- Traversing an implicit binary tree
- Nodes contain the mid-elements of the range under them
- At each node compare the desired object with the object at the node



Binary Search

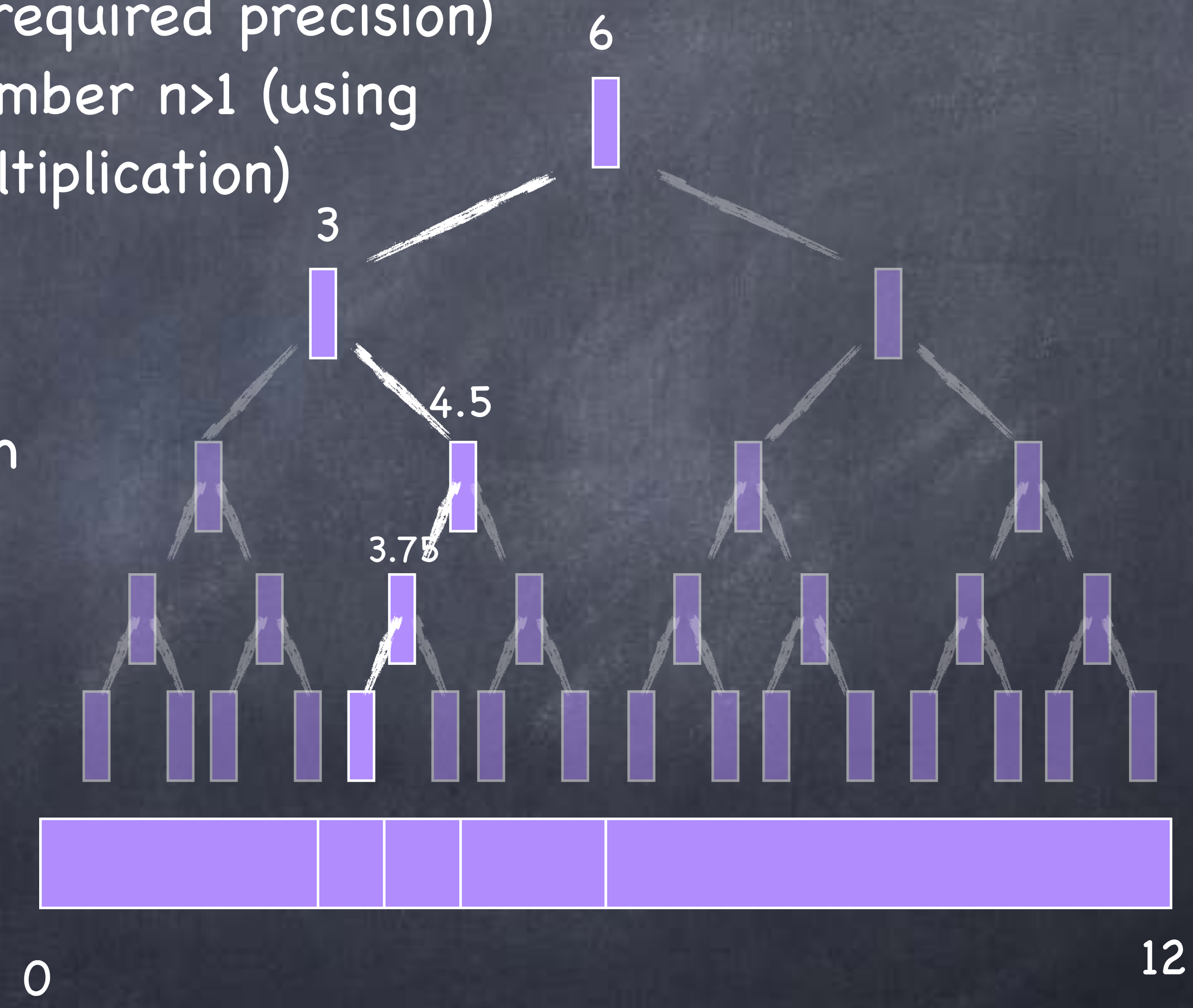
- Alternate use: to approximately find a root of a continuous function
  - Needs two points x_1, x_2, st. $f(x_1) \leq 0$ and $f(x_2) \geq 0$
  - Can maintain this invariant, while shrinking $|x_1-x_2|$ exponentially
  - Continuous $\longrightarrow$ this interval will have a root
  - May miss some 0s if function is not monotonous, but will find some other
  - Contrast with finding a 0 in an array of values [f(1),f(2),...,f(n)] (no continuity!)
    - If array not sorted, we may miss a 0, and there may not be another one!



- Faster methods exploit value/slope (not just sign)

Binary Search

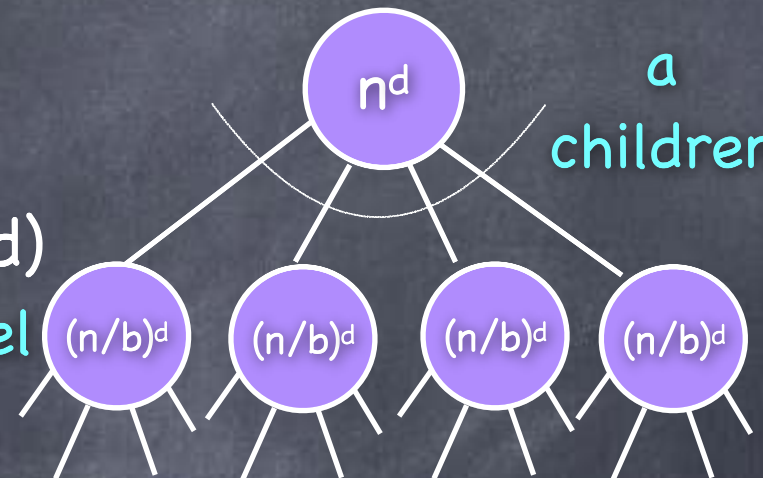
- Example: finding (up to required precision) the square root of a number n>1 (using only comparison and multiplication)
- Initial range: [0,n] (say)
- How to compare $\sqrt{n}$ with middle element m?
  - compare n and m^2



A General Solution (a.k.a. "Master Theorem")

- $T(n) = a\,T(n/b) + c \cdot n^d$ (and $T(1)=1$. $a \geq 1, b > 1$ integer, $c > 0$, $d \geq 0$ real.)
- Say $n = b^k$ (so only integers encountered)
- #levels $= \log_b n = k$

total at this level $= a \cdot (n/b)^d$



- $T(n) = O(\, n^d \,(\, 1 + (a/b^d) + \ldots + (a/b^d)^k \,)$ total at i^{th} level $= a^i \cdot (n/b^i)^d$
- If $a = b^d$, contribution at each level $= n^d$. $T(n) = O(n^d \cdot \log n)$
- If $a < b^d$: $1 + (a/b^d) + (a/b^d)^2 + \ldots = O(1)$. $T(n) = O(n^d)$
- If $a > b^d$: $(a/b^d)^k[1 + (b^d/a) + (b^d/a)^2 + \ldots\,] = O((a/b^d)^k) = a^k/n^d$
  $T(n) = O(a^k) = O(2^{k \cdot \log a}) = O(2^{\log n \,\cdot\, \log a/\log b}) = O(n^{\log_b a})$

Big Number Arithmetic

- Usually multiplication/addition are a single operation in a CPU
- But not possible when an integer has too many digits to fit into a processor's registers
- Can break up the integer into smaller pieces, and compute on them
  - e.g. Addition with carry: each operation (takes 2 numbers and a carry bit, and gives a number and a new carry bit) works on single digit numbers
  - To add two n-digit numbers: $O(n)$ operations
    - As fast as possible: need to at least read all the digits
    - (Remember: the number N has $n=O(\log N)$ digits)

Big Number Arithmetic

- Multiplication of two large (binary) numbers
  - First attempt: $x = x_0 + 2\,x_1$, where x_1 has one digit less Similarly, $y = y_0 + 2\,y_1$. So $x \cdot y = x_0y_0 + 2\,(x_0y_1 + x_1y_0) + 4x_1y_1$.
  - $T(n) = T(n-1) + O(n)$ (and $T(1)=O(1)$). So $T(n) = O(n^2)$
  - Can we do better by dividing the problem differently?
    - $x = x_0 + 2^{n/2}\,x_1$ where x_0, x_1 have n/2 digits each (assuming n is a power of 2)
    - $x \cdot y = x_0y_0 + 2^{n/2}(x_0y_1 + x_1y_0) + 2^nx_1y_1$, where all 4 products are of n/2 digit numbers (mult. by a power of 2 and addition take $O(n)$ time)
    - $T(n) = 4T(n/2) + \Theta(n)$. Still $T(n)=\Theta(n^2)$.
    - Can we do better?

Big Number Arithmetic

- Multiplication of two large numbers
  - $x = x_0 + 2^{n/2} x_1$ where x_0, x_1 have n/2 digits each (assuming n is a power of 2)
  - $x \cdot y = x_0 y_0 + 2^{n/2} (x_0 y_1 + x_1 y_0) + 2^n x_1 y_1$

    $= x_0 y_0 + 2^{n/2} [(x_0+x_1)(y_0+y_1) - x_0 y_0 - x_1 y_1] + 2^n x_1 y_1$

Karastuba's Algorithm

  - Only 3 multiplications (and reusing products). All of them on numbers about n/2 digits each
  - $T(n) = 3T(n/2) + O(n)$. $T(1) = O(1)$.
    - $a > b^d$, where $a=3$, $b=2$, $d=1$
    - $T(n) = O(n^{\log_2 3}) = O(n^{1.585..})$

Can do better, but more involved. Recently: O(n log n), but with a very large constant.

Fast Matrix Multiplication

- Multiplication of two large square matrices
  - Suppose we write A = $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, B = $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$
  - Then, AB = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, where $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j}$
  - Cost of multiplying two n×n matrices (assuming unit cost for both addition and multiplication)?
    - $T(n) = 8T(n/2) + cn^2$
    - $T(n) = n^{\log 8/\log 2} = n^3$
    - Same as the naïve algorithm, computing each of the n^2 terms of C using O(n) operations
- **Strassen's algorithm**: 7 smaller matrix multiplications instead of 8
- $T(n) = 7T(n/2) + cn^2 \Longrightarrow T(n) = O(n^{\log_2 7}) = O(n^{2.81})$

Can do better, but more involved.