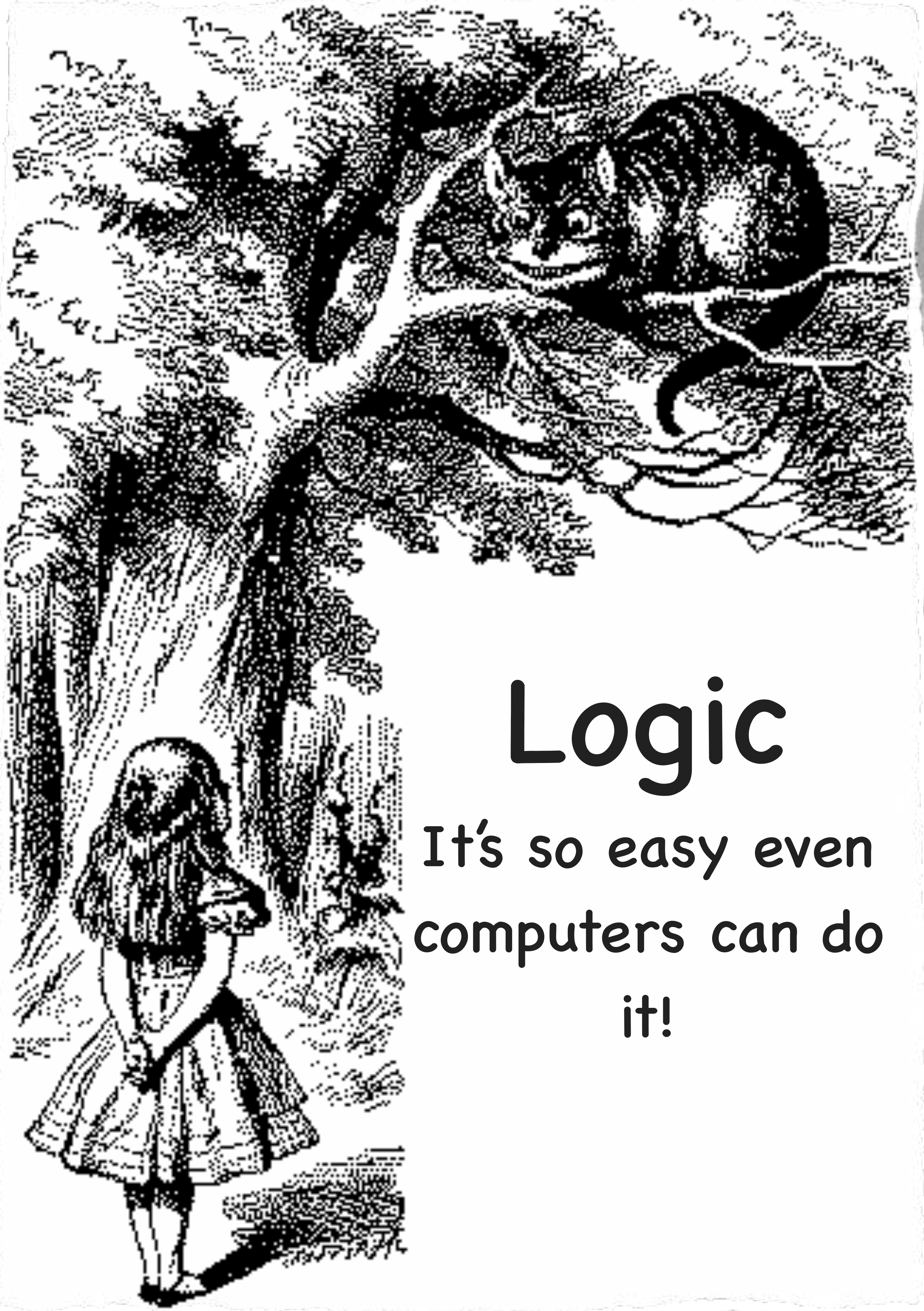
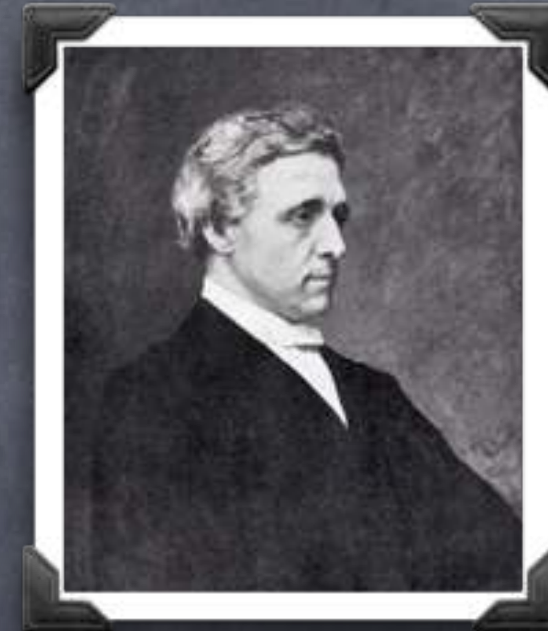


Logic

It's so easy even computers can do it!



Charles L Dodgson
1832 - 1898

Propositions, Predicates, Operators, Formulas

Logic in action

Expert Systems

- From a repository consisting of "facts" derive answers to questions posed on the fly
- To automate decision making
- e.g., Prolog: a programming language that can be used to implement such a system

```
mother_child(trude, sally).

father_child(tom, sally).
father_child(tom, erica).
father_child(mike, tom).

sibling(X, Y) ← parent_child(Z, X)
                ∧ parent_child(Z, Y).

parent_child(X, Y) ← father_child(X, Y).
parent_child(X, Y) ← mother_child(X, Y).
```

```
?- sibling(sally, erica).
Yes
```

The Pointless Game

- Alice and Bob sit down to play a new board game, where they take turns to make "moves" that they can choose (no dice/randomness)
- The rules of the game guarantee that
  - The game can't go on for ever
  - There are no ties — Alice or Bob will win when the game terminates

- Alice and Bob (smart as they are) decide that there is no point in playing the game, because they already know who is going to win it!

But how?

Propositions

- Goal: reasoning about whether statements are true or false
  - These statements are called **propositions**
- Propositions refer to things in a "**domain** of discourse" (e.g., characters in Alice in Wonderland)
- A proposition could simply refer to a property of an element in the domain (e.g., Alice doesn't have wings)
  - These properties are formalised as **predicates**

Alice
Jabberwock
Flamingo

Predicates

- Predicate is a function that assigns a value of TRUE or FALSE to each element in the domain of discourse
  - If you apply a predicate to an element you get a proposition
    - A proposition will have truth value True or False
  - More complex propositions can be built from such simple propositions

e.g.: Pink(Flamingo)

	Winged?	Flies?	Pink?
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

Propositional Calculus

cal·cu·lus /ˈkalkyələs/

1. The branch of mathematics that deals with the finding and properties of derivatives and integrals of functions, by methods originally
2. A particular method or system of calculation or reasoning.

Synonyms: stone - calculation - reckoning - computation

Unary operator

Binary operators

not p
Symbol: ¬p

Negates the truth value.

e.g.: ¬Flies(Alice) has value True

p or q
Symbol: p ∨ q

True if and only if at least one of p and q is true

e.g.: ¬Flies(Alice) ∨ Pink(Jab'wock) has value True

p and q
Symbol: p ∧ q

True if and only if both of p and q are true

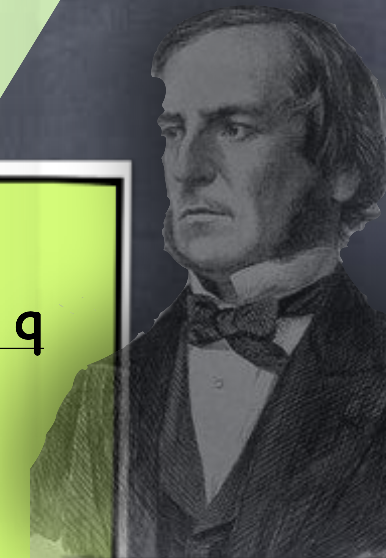
e.g.: ¬Flies(Alice) ∧ Pink(Jab'wock) has value False

if p then q
Symbol: p → q

(¬p) ∨ (p∧q)

Same as ¬p ∨ q

e.g.: Flies(Alice) → Pink(Jab'wock) has value True


George Boole
1815 - 1864

∨	T	F
T	T	T
F	T	F

∧	T	F
T	T	F
F	F	F

→	T	F
T	T	F
F	T	T

Operator Gallery

(T)	T	F
T	T	T
F	T	T

(F)	T	F
T	F	F
F	F	F

(1)	T	F
T	T	T
F	F	F

(2)	T	F
T	T	F
F	T	F

¬(1)	T	F
T	F	F
F	T	T

¬(2)	T	F
T	F	T
F	F	T

XOR ≠

⊕	T	F
T	F	T
F	T	F

IFF =

↔	T	F
T	T	F
F	F	T

OR

∨	T	F
T	T	T
F	T	F

NAND

↑	T	F
T	F	T
F	T	T

NOR

↓	T	F
T	F	F
F	F	T

AND

∧	T	F
T	T	F
F	F	F

IMPLIES

→	T	F
T	T	F
F	T	T

IMPLIED-BY

←	T	F
T	T	T
F	F	T

↛	T	F
T	F	T
F	F	F

↚	T	F
T	F	F
F	T	F

IMPLIES

→	T	F
T	T	F
F	T	T

p implies q.

whenever p holds, q holds

if p then q.

q if p.

either not p or (p and q).

p only if q.

if not q then not p.

not p if not q.

Important: Not a causal relation!

Try an example:

p: you're in the kitchen

q: you're in the house

Contrapositive

IMPLIED-BY

←	T	F
T	T	T
F	F	T

p is implied by q.

q implies p.

p if q.

IFF

=

↔	T	F
T	T	F
F	F	T

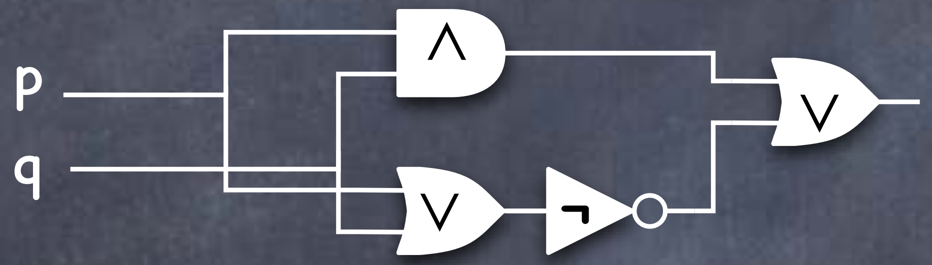
p if and only if q.

p iff q.

p if q and not p if not q.

Formulas

- A recipe for creating a new proposition from given propositions, using operators
  - e.g. $f(p,q) \triangleq (p \wedge q) \vee \neg(p \vee q)$



p	q	f
F	F	T
F	T	F
T	F	F
T	T	T

- Can also use "logic circuits" instead of formulas
- Different formulas can be equivalent to each other
  - e.g., $g(p,q) \triangleq \neg(p \oplus q)$. Then $f \equiv g$.
- A formula on two variables is <u>equivalent</u> to a binary operator

Another Example

- $g(p,q,r) \triangleq (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$
  - "Majority operator"
- $h(p,q,r) \triangleq (p \wedge (q \vee r)) \vee (q \wedge r)$
- $g \equiv h$

$((\neg p) \wedge q) \wedge r$

p	q	r	g	h
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
T	F	F	F	F
F	T	T	T	T
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

More Equivalences [Exercise]

- Conjunction and disjunction with T and F

$T \wedge q \equiv q$ | $F \vee q \equiv q$

$F \wedge q \equiv F$ | $T \vee q \equiv T$

- Implication involving T and F

$T \rightarrow q \equiv q$ | $F \rightarrow q \equiv T$

$q \rightarrow F \equiv \neg q$ | $q \rightarrow T \equiv T$

- Implication involving negation

$q \rightarrow \neg q \equiv \neg q$ | $\neg q \rightarrow q \equiv q$

- **Contrapositive**

$p \rightarrow q \equiv (\neg q) \rightarrow (\neg p)$

- **Distributive Property**

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **De Morgan's Law**

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$