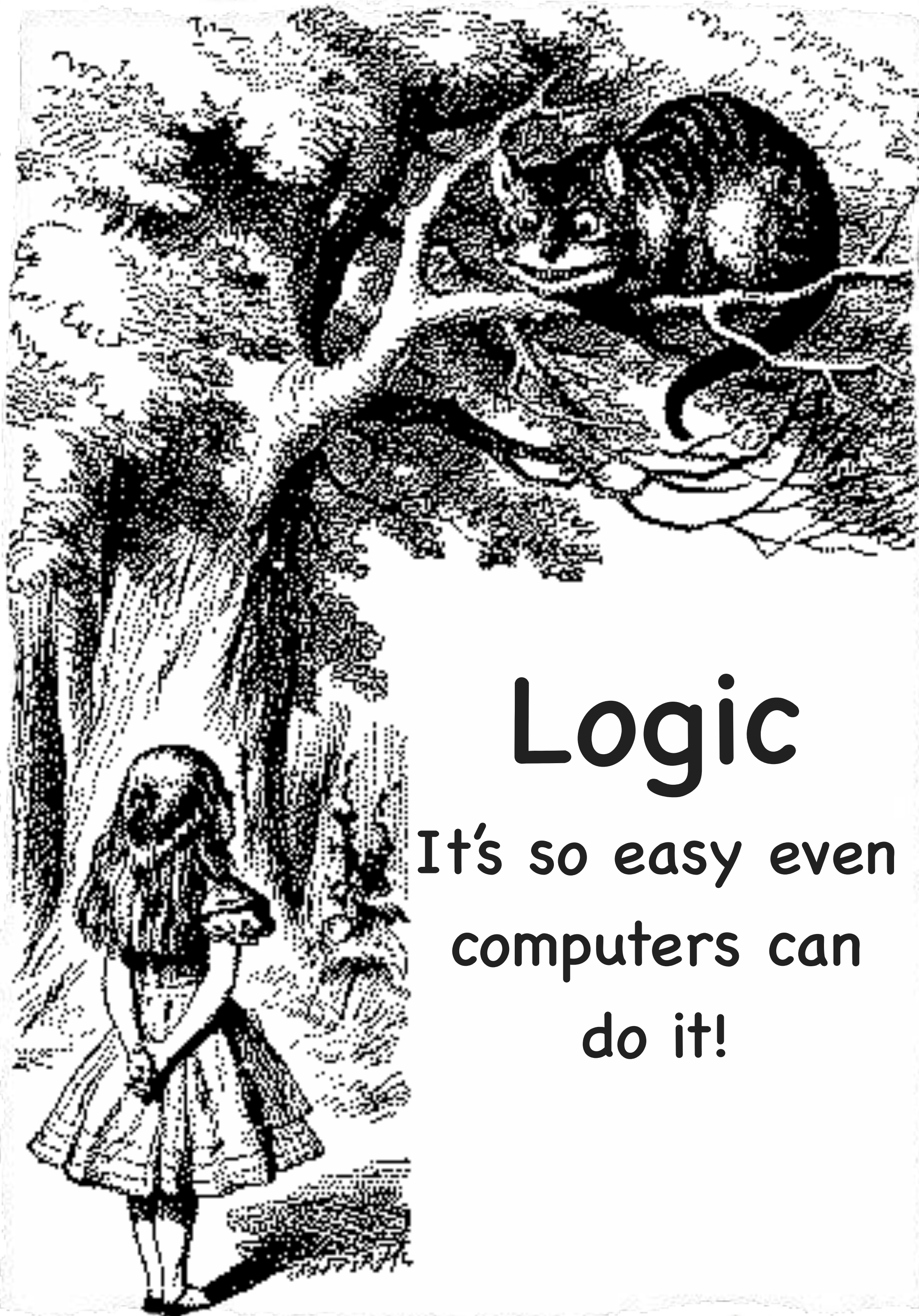
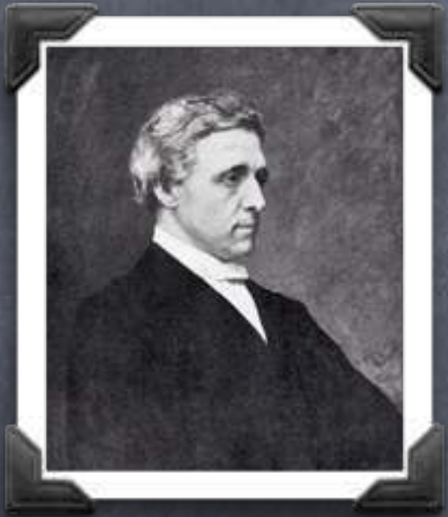


Logic

It's so easy even computers can do it!



Charles L Dodgson
1832 - 1898

Quantifiers

Predicates & Propositions

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- A predicate is a column in this table
- A proposition like Winged(Alice) refers to a single cell. Can build more complex propositions using propositional calculus (formulas)
- Next: Propositions involving quantifiers.

Quantified Propositions

(First-Order) Predicate Calculus

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

$\in$ AIW

- All characters in AIW are winged. (False!) — $\forall x$ Winged(x)
  - For every character x in AIW, Winged(x) holds
- Some character in AIW is winged. (True) — $\exists x$ Winged(x)
  - There exists a character x in AIW, such that Winged(x) holds

Quantified Propositions

(First-Order) Predicate Calculus

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- Quantifiers: To what "extent" does a predicate evaluate to TRUE in the domain of discourse
  - Universal quantifier, ∀
  - Existential quantifier, ∃

$\forall x$ Winged(x)

$\exists x$ Winged(x)

Quantified Propositions

(First-Order) Predicate Calculus

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- Could write $\forall x$ Winged(x) as:

  Winged(Alice) $\wedge$ Winged(J'wock) $\wedge$ Winged(Flamingo)

- And $\exists x$ Winged(x) as:

  Winged(Alice) $\vee$ Winged(J'wock) $\vee$ Winged(Flamingo)

  - But need to list the entire domain (works only if finite)

Examples

x	Winged(x)	Flies(x)	Pink(x)	Pink(x)→ Flies(x)
Alice	FALSE	FALSE	FALSE	TRUE
Jabberwock	TRUE	TRUE	FALSE	TRUE
Flamingo	TRUE	TRUE	TRUE	TRUE

- $\forall x \;\; \text{Winged}(x) \leftrightarrow \text{Flies}(x)$ is True
- $\exists x \;\; \text{Winged}(x) \rightarrow \neg\text{Flies}(x)$ is True
- $\forall x \;\; \text{Pink}(x) \rightarrow \text{Flies}(x)$ is True

Quantified Propositions

(First-Order) Predicate Calculus

x	Winged(x)	Flies(x)	Pink(x)	¬Winged(x)
Alice	FALSE	FALSE	FALSE	TRUE
Jabberwock	TRUE	TRUE	FALSE	FALSE
Flamingo	TRUE	TRUE	TRUE	FALSE

- $\forall x$ Winged(x) is False
  - Not everyone is winged
  - Same as saying, there is someone who is not winged
    - i.e., $\exists x$ ¬Winged(x) is True
- $\neg(\forall x\ W(x)) \equiv \exists x\ \neg W(x)$

$$\neg(W(a) \wedge W(j) \wedge W(f))$$
$$\equiv$$
$$\neg W(a) \vee \neg W(j) \vee \neg W(f)$$

Predicates, again

- A predicate can be defined over any number of elements from the domain
  - e.g., Likes(x,y): "x likes y"

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

Two quantifiers

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

- And we can quantify all the variables of a predicate
- e.g. $\forall x,y$ <u>Likes(x,y)</u>
  - Everyone likes everyone
  - False!

Two quantifiers

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

- $\forall x\ \exists y\ Likes(x,y)$
  - Everyone likes someone (True)
- $\exists y\ \forall x\ Likes(x,y)$
  - Someone is liked by everyone (False)

Order of quantifiers is important!

Two quantifiers

x	y	Likes(x,y)	∃y Likes(x,y) i.e., LikesSomeone(x)
Alice	Alice	TRUE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	
Jabberwock	Alice	FALSE	TRUE
	Jabberwock	TRUE	
	Flamingo	FALSE	
Flamingo	Alice	FALSE	TRUE
	Jabberwock	FALSE	
	Flamingo	TRUE	

- ∀x ∃y Likes(x,y)
  - Everyone likes someone
  - ∀x LikesSomeone(x)
  - True

Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$ for all pairs (x,y), P(x,y) holds
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$ for some pair (x,y), P(x,y) holds
- Below R is a proposition not involving x

  $\forall x\ P(x) \lor R \equiv (\forall x\ P(x)) \lor R$

- **Scope** of x extends to the end: $\forall x\ (P(x) \lor R)$
- i.e., if domain is $\{a_1,\ldots,a_N\}$ $(P(a_1)\lor R) \land \ldots \land (P(a_N)\lor R)$

- R evaluates to True or False (indep of x)
- When R is True, both equivalent (to True)
- Also, when R is False, both equivalent
- Hence both equivalent

Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$ for all pairs (x,y), P(x,y) holds
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$ for some pair (x,y), P(x,y) holds
- Below R is a proposition not involving x

  $\forall x\ P(x) \vee R \equiv (\forall x\ P(x)) \vee R$ $\quad$ $\exists x\ P(x) \vee R \equiv (\exists x\ P(x)) \vee R$

  $\forall x\ P(x) \wedge R \equiv (\forall x\ P(x)) \wedge R$ $\quad$ $\exists x\ P(x) \wedge R \equiv (\exists x\ P(x)) \wedge R$
- $\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$ $\quad$ $\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$
- $\forall x\ \underline{P(x) \rightarrow R} \equiv (\exists x\ P(x)) \rightarrow R$ $\quad$ $\exists x\ \underline{P(x) \rightarrow R} \equiv (\forall x\ P(x)) \rightarrow R$

$\forall x\ \underline{\neg P(x) \vee R} \equiv (\forall x\ \underline{\neg P(x)}) \vee R \equiv \neg\ (\exists x\ \underline{P(x)}) \vee R$

Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$ for all pairs (x,y), P(x,y) holds
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$ for some pair (x,y), P(x,y) holds
- Below R is a proposition not involving x

  $\forall x\ P(x) \lor R \equiv (\forall x\ P(x)) \lor R$ $\exists x\ P(x) \lor R \equiv (\exists x\ P(x)) \lor R$

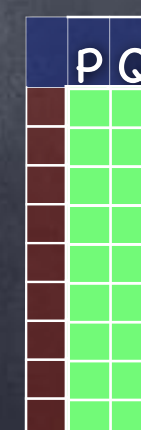
  $\forall x\ P(x) \land R \equiv (\forall x\ P(x)) \land R$ $\exists x\ P(x) \land R \equiv (\exists x\ P(x)) \land R$
- $\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$ $\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$
- $\forall x\ \underline{P(x) \rightarrow R} \equiv (\exists x\ P(x)) \rightarrow R$ $\exists x\ \underline{P(x) \rightarrow R} \equiv (\forall x\ P(x)) \rightarrow R$
- $(\forall x\ P(x)) \land (\forall x\ Q(x)) \equiv \forall x\ (P(x) \land Q(x))$

  $(\exists x\ P(x)) \lor (\exists x\ Q(x)) \equiv \exists x\ (P(x) \lor Q(x))$
- $(\forall x\ P(x)) \lor (\forall x\ Q(x)) \equiv (\forall x\ P(x)) \lor (\forall y\ Q(y))$

  $\equiv \forall x\ (\ P(x) \lor (\forall y\ Q(y))\ )$

  $\equiv \forall x\ (\ \forall y\ (P(x) \lor Q(y))\ )$

  $\equiv \forall x\ \forall y\ (P(x) \lor Q(y))$



Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$ for all pairs (x,y), P(x,y) holds
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$ for some pair (x,y), P(x,y) holds
- Below R is a proposition not involving x

  $\forall x\ P(x) \vee R \equiv (\forall x\ P(x)) \vee R$ $\exists x\ P(x) \vee R \equiv (\exists x\ P(x)) \vee R$

  $\forall x\ P(x) \wedge R \equiv (\forall x\ P(x)) \wedge R$ $\exists x\ P(x) \wedge R \equiv (\exists x\ P(x)) \wedge R$
- $\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$ $\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$
- $\forall x\ \underline{P(x) \rightarrow R} \equiv (\exists x\ P(x)) \rightarrow R$ $\exists x\ \underline{P(x) \rightarrow R} \equiv (\forall x\ P(x)) \rightarrow R$
- $(\forall x\ P(x)) \wedge (\forall x\ Q(x)) \equiv \forall x\ (P(x) \wedge Q(x))$

  $(\exists x\ P(x)) \vee (\exists x\ Q(x)) \equiv \exists x\ (P(x) \vee Q(x))$
- $(\forall x\ P(x)) \vee (\forall x\ Q(x)) \equiv \forall x\ \forall y\ P(x) \vee Q(y)$

  $(\exists x\ P(x)) \wedge (\exists x\ Q(x)) \equiv \exists x\ \exists y\ P(x) \wedge Q(y)$
- $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$ $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$

  $\nexists x\ P(x)$