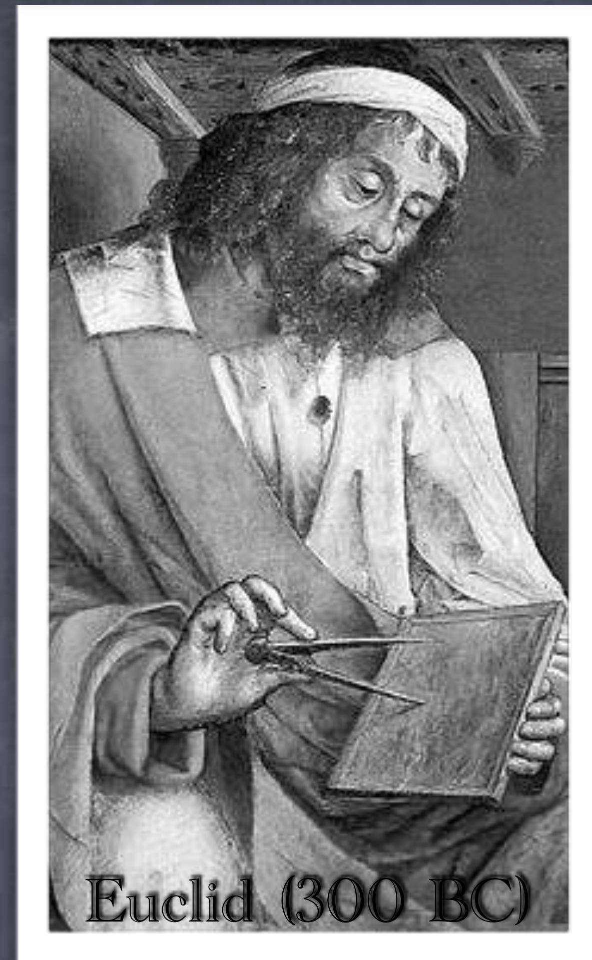

Mathematical Induction

Examples

Strong Induction

Induction hypothesis: $\forall n \le k\ P(n)$

- To prove $\forall n \in \mathbb{Z}^+\ P(n)$: we prove $P(1)$ (as before) and that

$$\forall k \in \mathbb{Z}^+ \quad (P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$$

$P(1)$	$P(1) \rightarrow P(2)$
$P(2)$	$P(1) \wedge P(2) \rightarrow P(3)$
$P(3)$	$P(1) \wedge .. \wedge P(3) \rightarrow P(4)$
$P(4)$	$P(1) \wedge .. \wedge P(4) \rightarrow P(5)$
$P(5)$	$P(1) \wedge .. \wedge P(5) \rightarrow P(6)$
:	:

Mathematical Induction

The fact that for any n, we can run this procedure to generate a proof for P(n), and hence for any n, P(n) holds.

$\forall n \in \mathbb{Z}^+\ P(n)$

Postage Stamps

- Claim: Every amount of postage that is at least ₹12 can be made from ₹4 and ₹5 stamps
  - i.e., $\forall n \in \mathbb{Z}^+ \;\; n \geq 12 \rightarrow \exists a,b \in \mathbb{N} \;\; n=4a+5b$
- Base cases: n=1,..,11 (vacuously true) and $n = 12 = 4\cdot 3 + 5\cdot 0$, $n = 13 = 4\cdot 2 + 5\cdot 1$, $n = 14 = 4\cdot 1 + 5\cdot 2$, $n = 15 = 4\cdot 0 + 5\cdot 3$.
- Induction step: For all integers $k \geq 16$:

  Strong induction hypothesis: Claim holds for all n s.t. $1 \leq n < k$

  To prove: Holds for $n=k$

  - $k \geq 16 \rightarrow k-4 \geq 12$.
  - So by induction hypothesis, $k-4=4a+5b$ for some $a,b \in \mathbb{N}$.
  - So $k = 4(a+1) + 5b$.

Prime Factorization

- Every positive integer $n \geq 2$ has a prime factorization i.e, $n = p_1 \cdot \ldots \cdot p_t$ (for some $t \geq 1$) where all p_i are prime

Need some more work to show unique factorization.

p prime $\wedge$ $p|ab$ $\rightarrow$ $p|a \vee p|b$

- Base case: $n=2$. ($t=1$, $p_1=2$).
- Induction step:

  (Strong) induction hypothesis: for all $n \leq k$, $\exists p_1,\ldots,p_t$, s.t. $n= p_1 \cdot \ldots \cdot p_t$

  To prove: $\exists q_1,\ldots,q_u$ (also primes) s.t. $k+1= q_1 \cdot \ldots \cdot q_u$

  - Case $k+1$ is prime: then $k+1=q_1$ for prime q_1
  - Case $k+1$ is not prime: $\exists a \in \mathbb{Z}^+$ s.t. $2 \leq a \leq k$ and $a|k+1$ (def. prime).
  - i.e., $\exists a,b \in \mathbb{Z}^+$ s.t. $2 \leq a,b \leq k$ and $k+1=a.b$ (def. divides; $a \geq 2 \rightarrow a.b > b$)
  - Now, by (strong) induction hypothesis, both a & b have prime factorizations: $a=p_1 \ldots p_s$, $b=r_1 \ldots r_t$.
  - Then $k+1=q_1 \ldots q_u$, where $u=s+t$, $q_i = p_i$ for $i=1$ to s and $q_i = r_{i-s}$, for $i=s+1$ to $s+t$.

Be careful about ranges!

- Claim: Every non-empty set of integers has either all elements even or all elements odd. (Of course, false!)
- "Proof" (bogus): By induction on the size of the set.
- Base case: $|S|=1$. The only element in S is either even or odd ✓
- Induction step: For all $k > 1$,
  Induction hypothesis: suppose all non-empty S with $|S| = k$, has either all elements even or all elements odd.
  To prove: then, it holds for all S with $|S|=k+1$.

Bug: Induction hypothesis cannot be bootstrapped from the base case

  - Let $S = \{a,b\} \cup S'$, where $|S'|=k-1$. (Note: S′ is not empty)
  - By IH, $S' \cup \{a\}$ has all even or all odd. Say, all even. Then S′ is all even. Now, $S' \cup \{b\}$ is also all even or all odd. Since S′ not empty, it is all even. Thus $S = S' \cup \{a,b\}$ is all even. QED.

Be careful about ranges!

- Claim: Every non-empty set of integers has either all elements even or all elements odd. (Of course, false!)
- "Proof" (bogus): By induction on the size of the set.
    - We proved $P(1)$ and $\forall k>1 \quad P(k) \rightarrow P(k+1)$

$P(1)$	
	$P(2) \rightarrow P(3)$
	$P(3) \rightarrow P(4)$
	$P(4) \rightarrow P(5)$
	$P(5) \rightarrow P(6)$
	$\vdots$

Nim

- Alice and Bob take turns removing matchsticks from two piles
- Initially both piles have equal number of matchsticks
- At every turn, a player must choose one pile and remove <u>one or more</u> matchsticks from that pile
- Goal: be the person to remove the last matchstick

- Claim: In Nim, the second player has a <u>winning strategy</u>
  - (Aside: in <u>every</u> finitely-terminating two player game without draws, one of the players has a winning strategy)

- Claim: The following is a winning strategy for the second player: keep the piles matched at the end of your turn

Nim

- Claim: The following is a winning strategy for the second player: keep the piles matched at the end of your turn
- Induction variable: n = number of matchsticks on each pile at the beginning of the game.
- Base case: $n=1$. Alice must remove one. Next, Bob wins. ✔
- Induction step: for all integers $k \geq 1$

  Induction hypothesis: when starting with $n \leq k$, Bob always wins

  strong

  To prove: when starting with $n=k+1$, Bob always wins
  - Case 1: Alice removes all $k+1$ from one pile. Next, Bob wins.
  - Case 2: Alice removes j, $1 \leq j \leq k$ from one pile. After Bob's move $k+1-j$ left in each pile. By induction hypothesis, Bob will win from here.