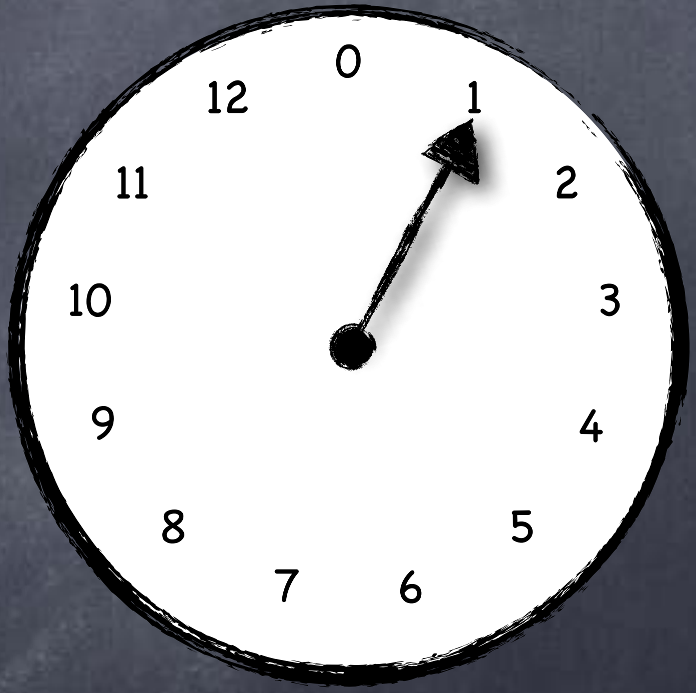
Numb3rs

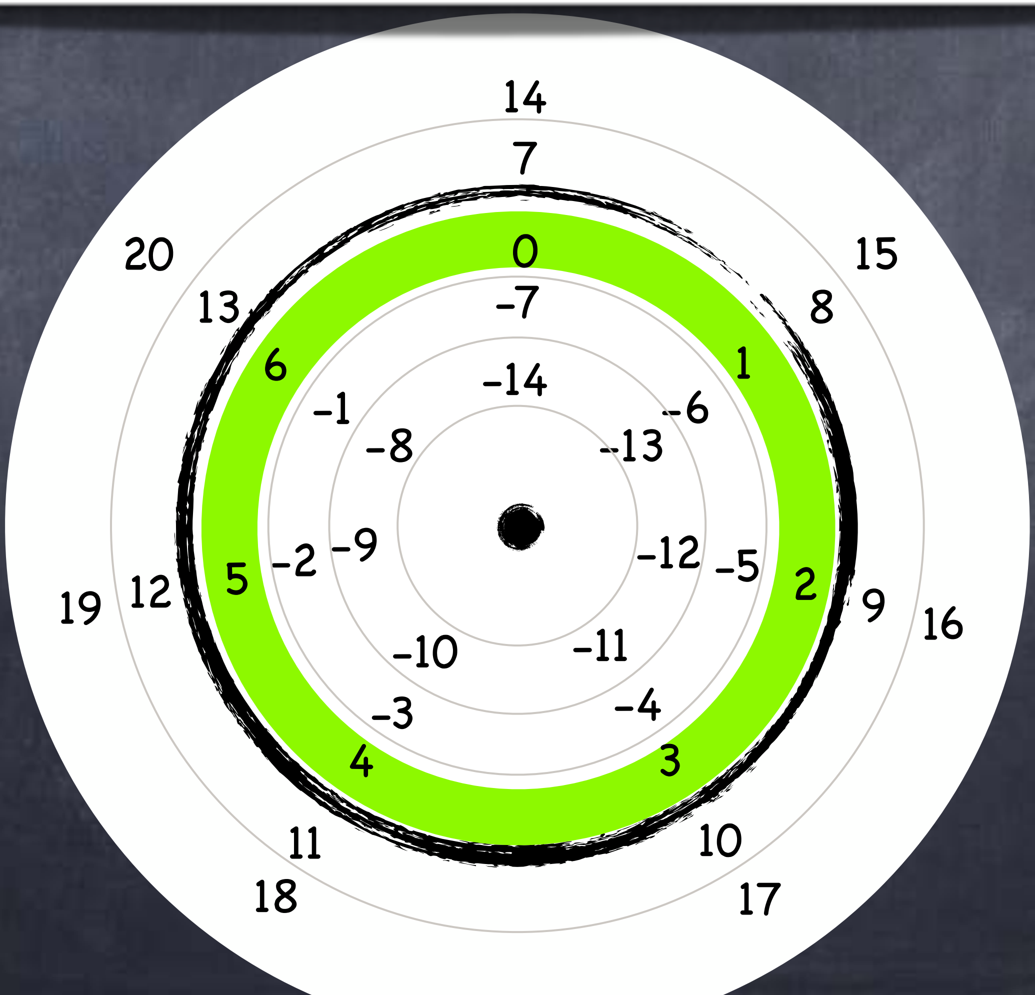
The Skippy Clock



Congruence Classes

$[a]_m = \{ x \mid a \equiv x \pmod{m} \}$

can be represented by rem(a,m)



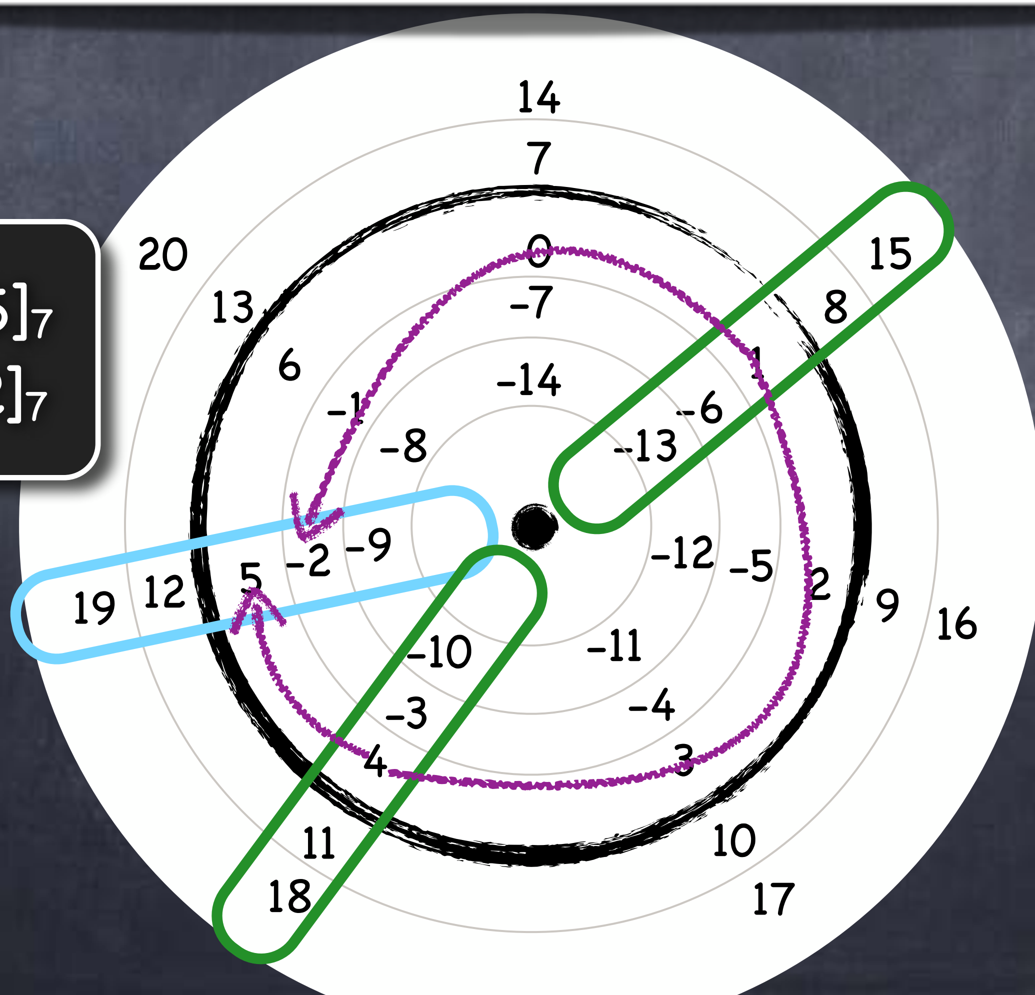
m=7

Modular Addition

Modular addition: $[a]_m +_m [b]_m = [a+b]_m$

$[4]_7 + [1]_7 = [5]_7$

$[4]_7 + [-6]_7 = [-2]_7$

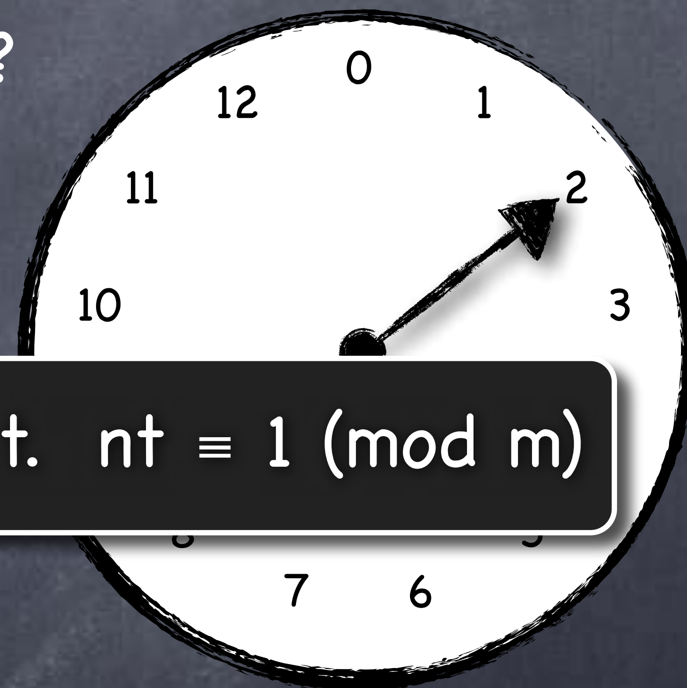


m=7

The Skippy Clock

- Has m hours on its dial, needle moves n hours at a time
- Will the needle reach all the hours?
- Iff needle reaches 1 (why?)
  - i.e., $\exists t \geq 0$ s.t. $nt \equiv 1 \pmod{m}$
- Iff $\gcd(m,n) = 1$
  - $\gcd(m,n)=1 \leftrightarrow n^{-1}$ exists in $\mathbb{Z}_m$
  - $\gcd(m,n)=1 \leftrightarrow \exists u,v \; mu+nv=1$

    $\leftrightarrow \exists v \; [n]_m \times_m [v]_m = [1]_m$



$t \in [0,m)$ s.t. $nt \equiv 1 \pmod{m}$

The Skippy Clock

- Has m hours on its dial, needle moves n hours at a time
- Will the needle reach all the hours?
  - Iff gcd(m,n) = 1
    - Then how long will it take to reach all the hours?
  - If gcd(m,n)≠1, how many hours are reached?
  - What is the first hour that gets repeated?
  - Look at the multiplication table of $\mathbb{Z}_m$!

Multiplication Table

- Has m hours on its dial, needle moves n hours at a time

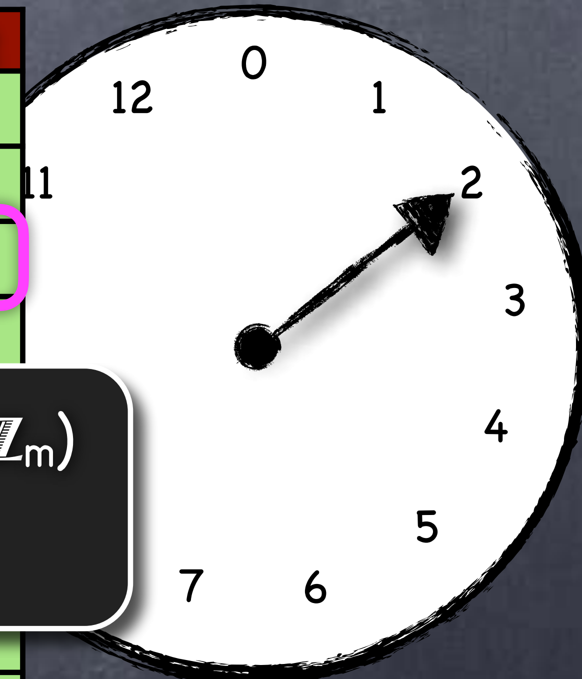
×	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	1	3	5	7	9	11
3	0	3	6	9	12	2	5	8	11	1	4	7	10
4	0	4	8	12	3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	0	5	10	2	7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6	0	6	12	5	11	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7	0	7	1	8	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5	12	6
8	0	8	3	11	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	10	5
9	0	9	5	1	10	6	2	11	7	3	12	8	4
10	0	10	7	4	1	11	8	5	2	12	9	6	3
11	0	11	9	7	5	3	1	12	10	8	6	4	2
12	0	12	11	10	9	8	7	6	5	4	3	2	1

$\gcd(m,n)=1 \leftrightarrow n^{-1}$ exists (in $\mathbb{Z}_m$)

$\leftrightarrow \forall z\ \exists t\ nt=z$ (in $\mathbb{Z}_m$)

take z=1

$t = n^{-1} z$



m=13

The Skippy Clock

- Has m hours on its dial, needle moves n hours at a time
- Will the needle reach all the hours?
  - Iff gcd(m,n) = 1
    - Then how long will it take to reach all the hours?

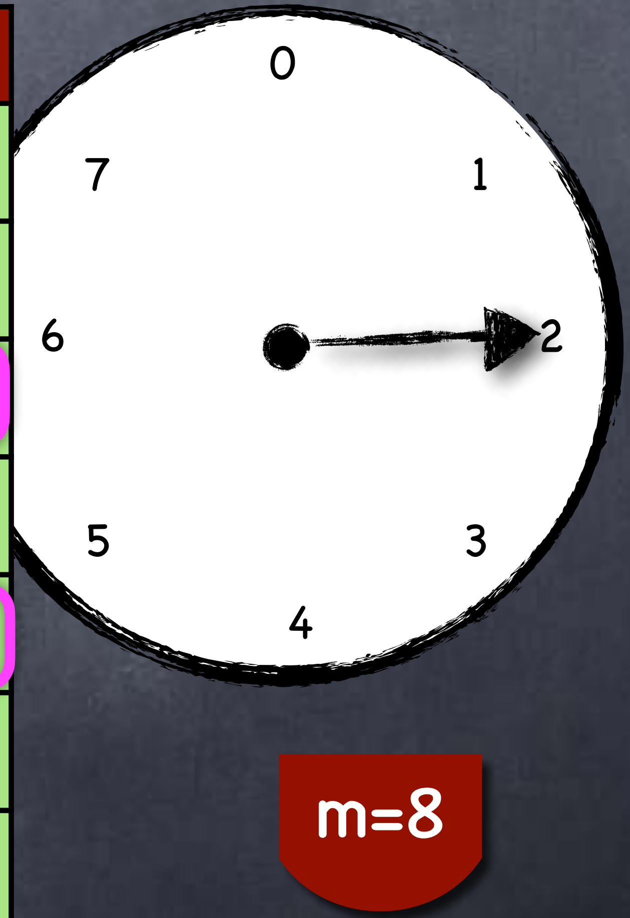
m-1 steps

  - If gcd(m,n)≠1, how many hours are reached?
  - What is the first hour that gets repeated?

With Common Factors

- Has m hours on its dial, needle moves n hours at a time

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1



With Common Factors

- Has m hours on its dial, needle moves n hours at a time
- Say, gcd(m,n) = g
- m = gp, n = gq, where gcd(p,q)=1
- Consider a new clock, with p hours on the dial and needle moving q hours at a time

- The original clock can be obtained by sub-dividing each hour in the new clock into g hours
- New clock: needle reaches all p hours {0,1,...,p-1}

  ⇒ Original clock: it reaches p hours, {0,g,...g(p-1)}

The Skippy Clock

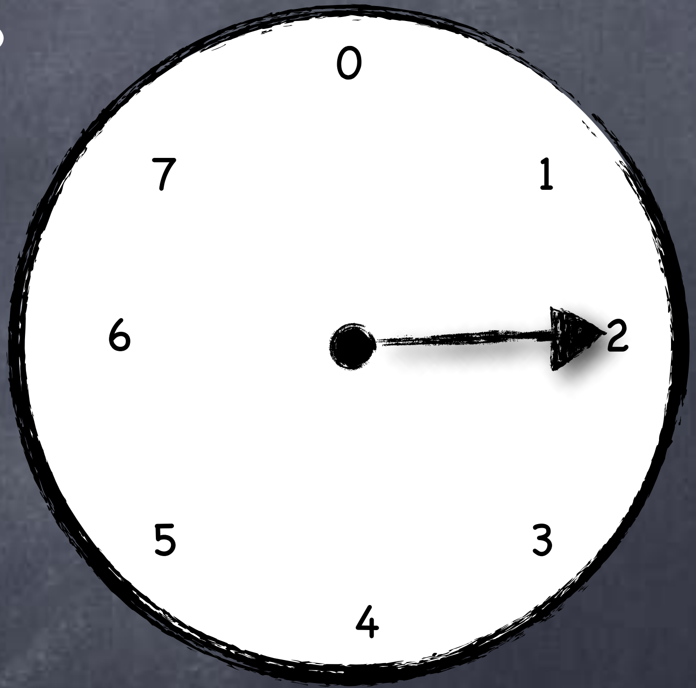
- Has m hours on its dial, needle moves n hours at a time
- Will the needle reach all the hours?
  - Iff gcd(m,n) = 1
    - Then how long will it take to reach all the hours?

m-1 steps

  - If gcd(m,n)≠1, how many hours are reached?

m/gcd(m,n)

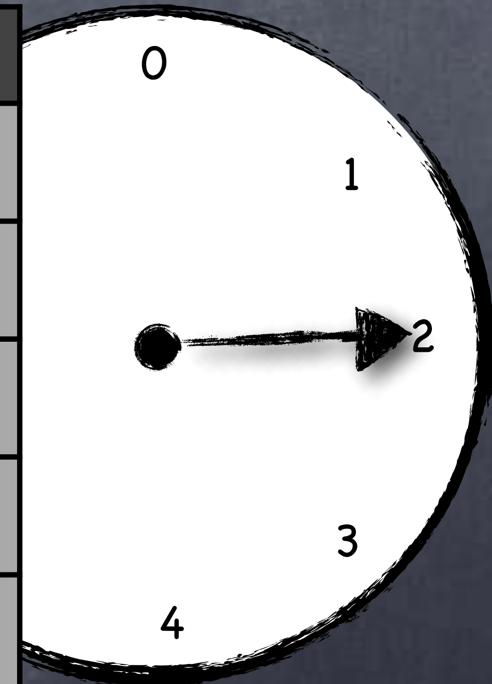
  - What is the first hour that gets repeated?



Repeating

- Has m hours on its dial, needle moves n hours at a time

×	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	0
2	0	2	4	6	0	2	4	6	0
3	0	3	6	1	4	7	2	5	0
4	0	4	0	4	0	4	0	4	0
5	0	5	2	7	4	1	6	3	0
6	0	6	4	2	0	6	4	2	0
7	0	7	6	5	4	3	2	1	0



m=8

The Skippy Clock

- Has m hours on its dial, needle moves n hours at a time
- Will the needle reach all the hours?
    - Iff gcd(m,n) = 1
        - Then how long will it take to reach all the hours?
    - If gcd(m,n)≠1, how many hours are reached?
    - What is the first hour that gets repeated?

m-1 steps

m/gcd(m,n)

0