Sets & Relations

Basics of Sets

Sets & Relations in action

Relational Database

x	y	Likes(x,y)
Alice	Alice	TRUE
	Jabberwock	FALSE
	Flamingo	TRUE
Jabberwock	Alice	FALSE
	Jabberwock	TRUE
	Flamingo	FALSE
Flamingo	Alice	FALSE
	Jabberwock	FALSE
	Flamingo	TRUE

Relational DB Table

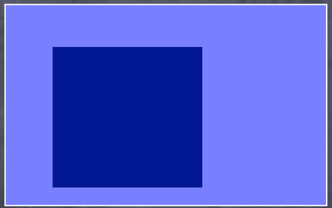
Likes	
x	y
Alice	Alice
Alice	Flamingo
Jabberwock	Jabberwock
Flamingo	Flamingo

- Queries to the DB are set/logical operations
  - SELECT x
    FROM Likes
    WHERE y='Alice' OR y='Flamingo'
  - $\{ x \mid (x, \text{Alice}) \in \text{Likes} \} \cup \{ x \mid (x, \text{Flamingo}) \in \text{Likes} \}$

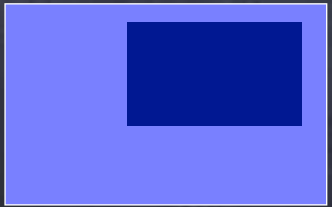
Sets: Basics

- Unordered collection of "elements"
  - e.g.: $\mathbb{Z}$, $\mathbb{R}$ (infinite sets), $\emptyset$ (empty set), {1, 2, 5}, ...
- Will always be given an implicit or explicit universe (universal set) from which the elements come
  - (Aside: In developing the foundations of mathematics, often one starts from "scratch", using only set theory to create the elements themselves)
- Set membership: e.g. $0.5 \in \mathbb{R}$, $0.5 \notin \mathbb{Z}$, $\emptyset \notin \mathbb{Z}$
- Set inclusion: e.g. $\mathbb{Z}$, $\subseteq \mathbb{R}$, $\emptyset \subseteq \mathbb{Z}$
- Set operations: complement, union, intersection, difference

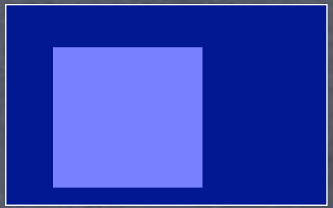
Set Operations



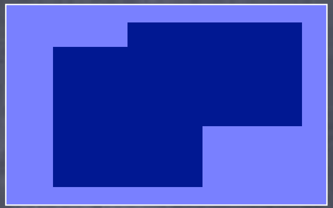
S



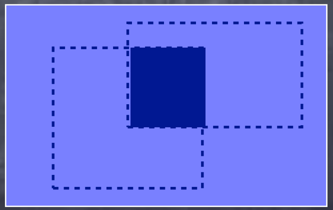
T



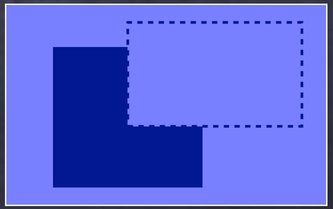
$\overline{S}$



$S \cup T$



$S \cap T$



$S - T$

Sets as Predicates

x	Winged(x)	Flies(x)	Pink(x)	inClub(x)
Alice	FALSE	FALSE	FALSE	TRUE
Jabberwock	TRUE	TRUE	FALSE	FALSE
Flamingo	TRUE	TRUE	TRUE	TRUE

- Given a predicate can define the set of elements for which it holds
  - WingedSet = { x | Winged(x) } = {J'wock, Flamingo}
  - FliesSet = { x | Flies(x) } = {J'wock, Flamingo}
  - PinkSet = { x | Pink(x) } = {Flamingo}

- Conversely, given a set, can define a membership predicate for it e.g. given set Club = {Alice, Flamingo}. Then, define predicate inClub(x) s.t. inClub(x) = True iff $x \in$ Club

Set Operations

Unary operator

Binary operators

Associative

S **complement**	S **union** T	S **intersection** T	S **difference** T	S **symmetric diff.** T
Symbol: $\overline{S}$	Symbol: $S \cup T$	Symbol: $S \cap T$	Symbol: $S - T$ (Alternately: $S \setminus T$)	Symbol: $S \Delta T$
$in\overline{S}(x) \equiv \neg inS(x)$	$inS{\cup}T(x) \equiv inS(x) \lor inT(x)$	$inS{\cap}T(x) \equiv inS(x) \land inT(x)$	$inS{-}T(x) \equiv inS(x) \land \neg inT(x) \equiv inS(x) \nleftarrow inT(x))$ $S-T = S \cap \overline{T}$	$inS\Delta T(x) \equiv inS(x) \oplus inT(x)$

Note: Notation inS(x) used only to explicate the connection with predicate logic. Will always write $x \in S$ later.

De Morgan's Laws

- $\overline{S \cup T} = \overline{S} \cap \overline{T}$
  - $x \in \overline{S \cup T} \equiv \neg(x \in S \cup T)$

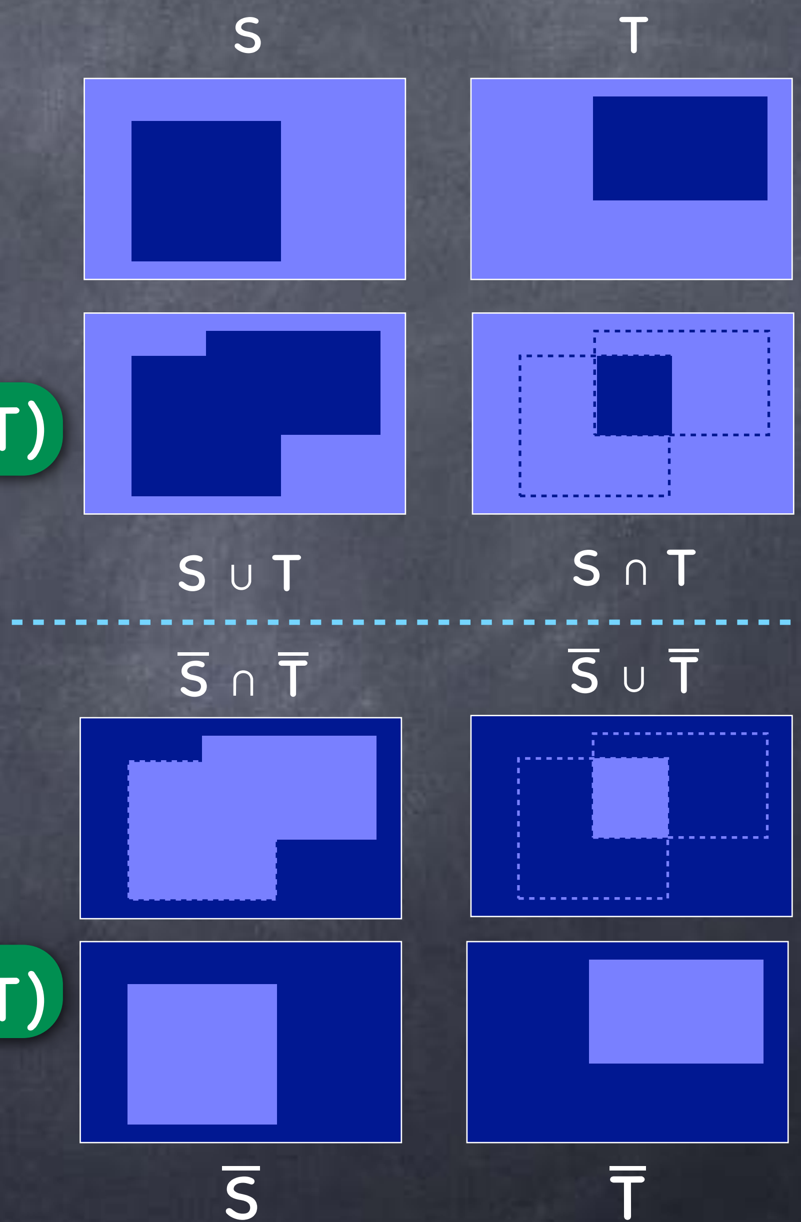
    $\equiv \neg(x \in S \lor x \in T) \equiv \neg(x \in S) \land \neg(x \in T)$

    $\equiv x \in \overline{S} \land x \in \overline{T} \equiv x \in \overline{S} \cap \overline{T}$

- $\overline{S \cap T} = \overline{S} \cup \overline{T}$
  - $x \in \overline{S \cap T} \equiv \neg(x \in S \cap T)$

    $\equiv \neg(x \in S \land x \in T) \equiv \neg(x \in S) \lor \neg(x \in T)$

    $\equiv x \in \overline{S} \lor x \in \overline{T} \equiv x \in \overline{S} \cup \overline{T}$



Distributivity

- $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$
  - $x \in R \cap (S \cup T) \equiv$

    $\equiv x \in R \wedge (x \in S \vee x \in T) \equiv (x \in R \wedge x \in S) \vee (x \in R \wedge x \in T)$

    $\equiv x \in (R \cap S) \cup (R \cap T)$
- $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
  - $x \in R \cup (S \cap T) \equiv$

    $\equiv x \in R \vee (x \in S \wedge x \in T) \equiv (x \in R \vee x \in S) \wedge (x \in R \vee x \in T)$

    $\equiv x \in (R \cup S) \cap (R \cup T)$

Set Inclusion

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- PinkSet $\subseteq$ FliesSet = WingedSet

- S $\subseteq$ T same as the proposition $\underline{\forall x \;\; x \in S \rightarrow x \in T}$
- S $\supseteq$ T same as the proposition $\underline{\forall x \;\; x \in S \leftarrow x \in T}$
- S = T same as the proposition $\underline{\forall x \;\; x \in S \leftrightarrow x \in T}$

Set Inclusion

- $S \subseteq T$ same as the proposition $\underline{\forall x \;\; x \in S \rightarrow x \in T}$
- If $S = \emptyset$, and T any arbitrary set, $S \subseteq T$
  - $\forall x$, vacuously we have $x \in S \rightarrow x \in T$
- If $S \subseteq T$ and $T \subseteq R$, then $S \subseteq R$
  - Consider arbitrary $x \in S$. Since $S \subseteq T$, $x \in T$. Then since $T \subseteq R$, $x \in R$.

If no such x, already done

- $S \subseteq T \longleftrightarrow \overline{T} \subseteq \overline{S}$
  - $\forall x \;\; \underline{x \in S \rightarrow x \in T} \;\; \equiv \;\; \forall x \;\; \underline{x \notin T \rightarrow x \notin S}$ (contrapositive)

$$\equiv \;\; \forall x \;\; \underline{x \in \overline{T} \rightarrow x \in \overline{S}}$$

Proving Set Equality

- To prove S = T, show $S \subseteq T$ and $T \subseteq S$
- e.g., $L(a,b) = \{ x : \exists u,v \in \mathbb{Z} \;\; x=au+bv \}$

  $M(a,b) = \{ x : (\gcd(a,b) \mid x) \}$
- [Recall] **Theorem**: $L(a,b) = M(a,b)$
- Proof in two parts:
  - $L(a,b) \subseteq M(a,b)$: i.e., $\forall x \in \mathbb{Z} \;\; x \in L(a,b) \rightarrow x \in M(a,b)$
  - $M(a,b) \subseteq L(a,b)$: i.e., $\forall x \in \mathbb{Z} \;\; x \in M(a,b) \rightarrow x \in L(a,b)$

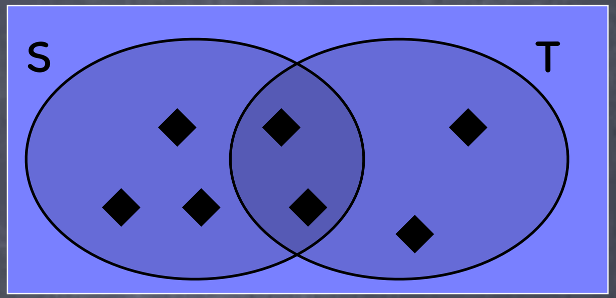
Let $x=au+bv$.
$g|a, g|b \Rightarrow g|x$

First show that
$g \in L(a,b)$
(as the smallest +ve element in L(a,b))

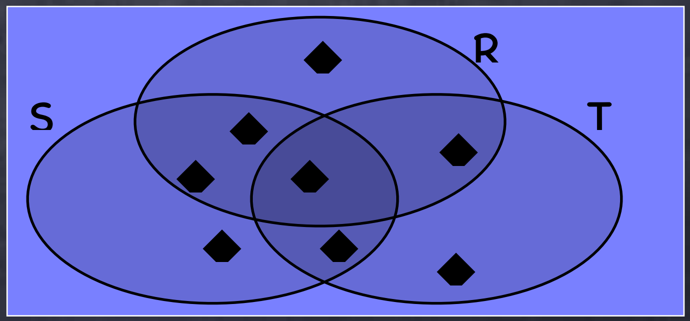
Let $x=ng$. But
$g=au+bv \Rightarrow x=au'+bv'$

Inclusion-Exclusion

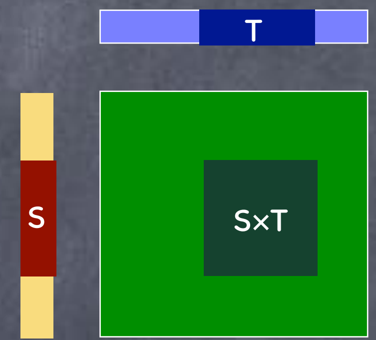
- $|S| + |T|$ counts every element that is in S or in T
  - But it double counts the number of elements that are in both: i.e., elements in $S \cap T$
- So, $|S|+|T| = |S \cup T| + |S \cap T|$
- Or, $|S \cup T| = |S| + |T| - |S \cap T|$



- $|R \cup S \cup T| = |R|+|S|+|T| - |R \cap S| - |S \cap T| - |T \cap R| + |R \cap S \cap T|$
  - $|R \cup S \cup T| = |R| + |S \cup T| - |R \cap (S \cup T)|$
    $= |R| + |S \cup T| - |(R \cap S) \cup (R \cap T)|$
    $= |R| + |S| + |T| - |S \cap T|$
    $-(|R \cap S| + |R \cap T| - |R \cap S \cap T|)$



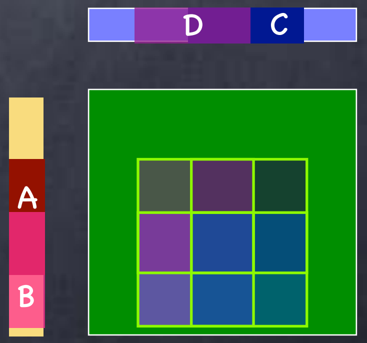
Cartesian Product



- $S \times T = \{ (s,t) \mid s \in S \text{ and } t \in T \}$
  - $(S = \emptyset \vee T = \emptyset) \leftrightarrow S \times T = \emptyset$
  - $|S \times T| = |S| \cdot |T|$
- $R \times S \times T = \{ (r,s,t) \mid r \in R, s \in S, t \in T \}$
  - Not the same as $(R \times S) \times T$ (but "essentially" the same)

$\{ ((r,s),t) \mid r \in R, s \in S, t \in T \}$

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$. Also, $(A \cap B) \times C = (A \times C) \cap (B \times C)$
  - $(A \cup B) \times (C \cup D) = (A \times (C \cup D)) \cup (B \times (C \cup D))$
    $= (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$



- Complement: $\overline{S \times T} = ?$
  - $(\overline{S} \times \overline{T}) \cup (\overline{S} \times T) \cup (S \times \overline{T})$