Sets & Relations

Relations

Relations: Basics

More commonly written as:
x Likes y, $x \sqsubset y$, $x \geq y$, $x \sim y$, xLy, ...

- Informally, a relation is specified as what is related to what
- Formally, a predicate over the domain $S \times S$
  - e.g. Likes(x,y)
- Alternately, a subset of $S \times S$, namely the pairs for which the relation holds
  - Likes = { (Alice,Alice), (Alice, Flamingo), (J'wock,J'wock), (Flamingo,Flamingo) }

Homogeneous and binary (the default notion for us)

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

Many ways to look at it!

$R \subseteq S \times S$

a set of ordered-pairs

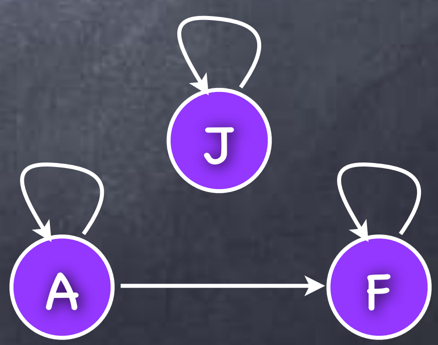
$\{ (a,b) \mid a \sqsubset b \}$

{ (A,A), (A,F), (J,J), (F,F) }

Boolean matrix, $M_{a,b}$ = T iff $a \sqsubset b$

	A	J	F
A	T	F	T
J	F	T	F
F	F	F	T

(directed) graph

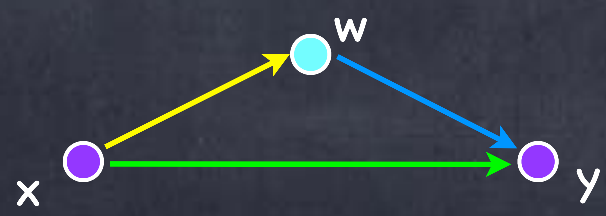


Operations on Relations

- Since a relation is a set, namely $R \subseteq S \times S$, all set operations extend to relations
  - Complement (with the universe being $S \times S$), Union, Intersection, Symmetric Difference
- Converse (a.k.a. Transpose)

$M^T_{xy} = M_{yx}$

  - $R^T = \{ (x,y) \mid (y,x) \in R \}$
- Composition

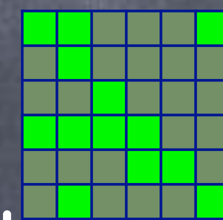


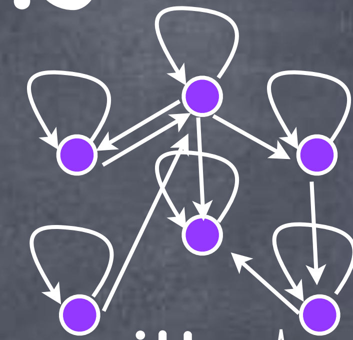
"Boolean matrix multiplication"
$(M \circ M')_{xy} = \bigvee_w (M_{xw} \wedge M'_{wy})$

  - $R \circ R' = \{ (x,y) \mid \exists w \in S \ \ (x,w) \in R \text{ and } (w,y) \in R' \}$

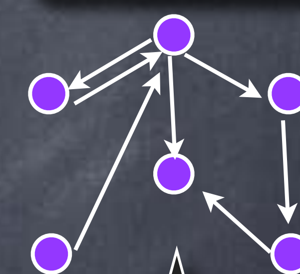
(Ir)Reflexive Relations

- Reflexive (e.g. Knows, ≤)
  - The kind of relationship that everyone has with themselves





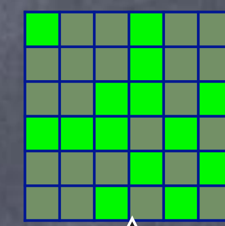
- Irreflexive (e.g. Gave birth to, ≠)
  - The kind that nobody has with themselves



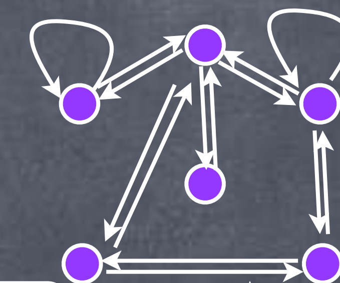
- Neither (e.g. is a prime factor of)
  - Some, but not all, have this relationship with themselves

(Anti)Symmetric Relations

- Symmetric (e.g. sits next to)
  - The relationship is reciprocated



symmetric matrix



self-loops & bidirectional edges only

- Anti-symmetric (e.g. parent of, prime factor of, ⊆)
  - No reciprocation (except possibly with self)

no bidirectional edges

- Neither (e.g. likes)
  - Reciprocated in some pairs (with distinct members) and only one-way in other pairs

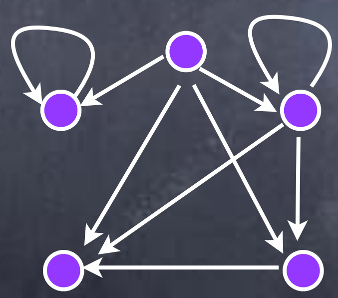
some bidirectional, some unidirectional

- Both (e.g., =)
  - Each one related only to self (if at all)

no edges except self-loops

Transitive Relations

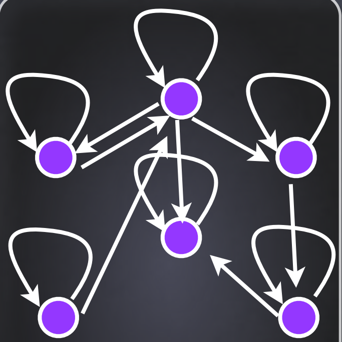
- Transitive (e.g., Ancestor of, subset of, divides, $\leq$)
  - if a is related to b and b is related to c, then a is related to c
- R is transitive $\leftrightarrow$ $R \circ R \subseteq R$ $\leftrightarrow$ $\forall k{>}1\ R^k \subseteq R$



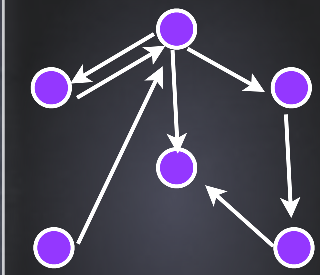
> if there is a "path" from a to z, then there is edge (a,z)

- Intransitive: Not transitive

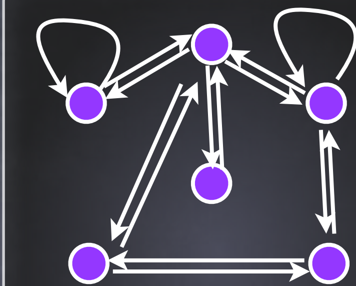
Types of Relations



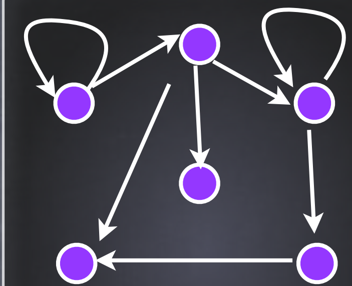
Reflexive: All self-loops



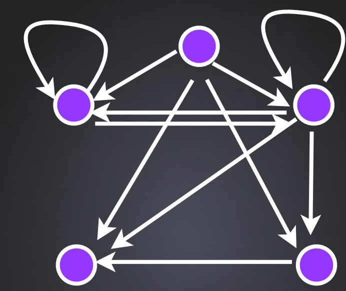
Irreflexive: No self-loops



Symmetric: Only self-loops & bidirectional edges



Anti-symmetric: No bidirectional edges



Transitive: Path from a to b implies edge (a,b)

The complete relation $R = S \times S$ is reflexive, symmetric and transitive

Reflexive closure of R: Minimal relation $R' \supseteq R$ s.t. R' is reflexive

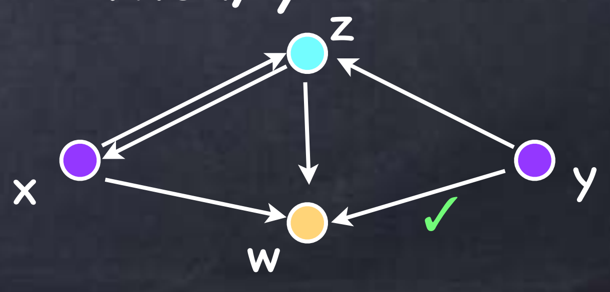
Symmetric closure of R: Minimal relation $R' \supseteq R$ s.t. R' is symmetric

Transitive closure of R: Minimal relation $R' \supseteq R$ s.t. R' is transitive

Each of these is unique [Why?]

Equivalence Relation

- A relation that is reflexive, symmetric and transitive
  - e.g. is a relative, has the same last digit, is congruent mod 7, ...
- Equivalence class of x: **$Eq(x) \triangleq \{y | x \sim y\}$**.
- Every element is in its own equivalence class (by reflexivity)
- Claim: If $Eq(x) \cap Eq(y) \neq \emptyset$, then $Eq(x) = Eq(y)$.
  - Let $z \in Eq(x) \cap Eq(y)$. To show $Eq(x) \subseteq Eq(y)$ [similarly, $Eq(y) \subseteq Eq(x)$]
  - Consider an arbitrary $w \in Eq(x)$: i.e., $x \sim w$.
  - By symmetry, $z \sim x$. Then, by transitivity, $z \sim w$. Then, $y \sim w$.
  - Thus, $w \in Eq(y)$. i.e., $Eq(x) \subseteq Eq(y)$.



Equivalence Relation

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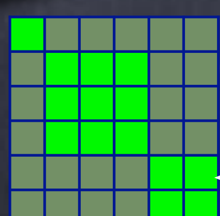
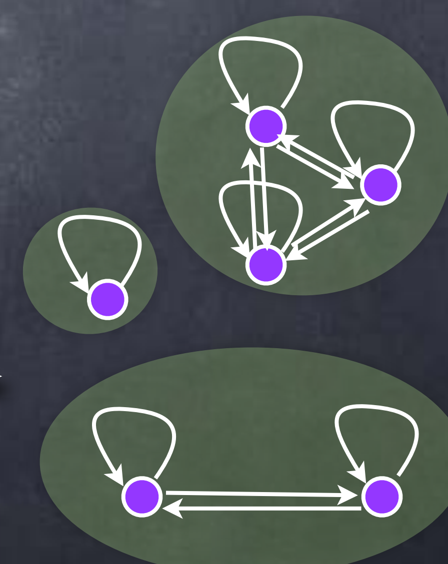
- Every element is in its own equivalence class (by reflexivity)
- Claim: If $Eq(x) \cap Eq(y) \neq \emptyset$, then $Eq(x) = Eq(y)$.
- The equivalence classes partition the domain

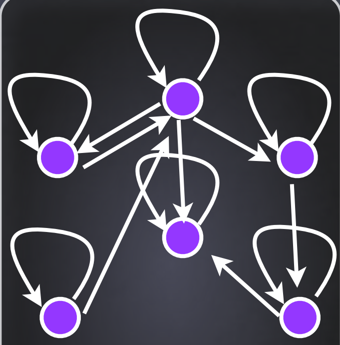
$P_1,..,P_t \subseteq S$
s.t.
$P_1 \cup .. \cup P_t = S$
$P_i \cap P_j = \emptyset$



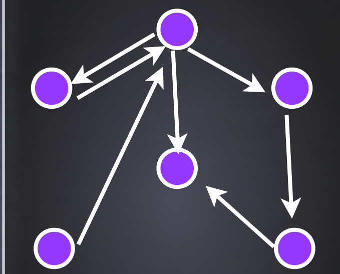
Square blocks along the diagonal, after sorting the elements by equivalence class

"Cliques" for each class

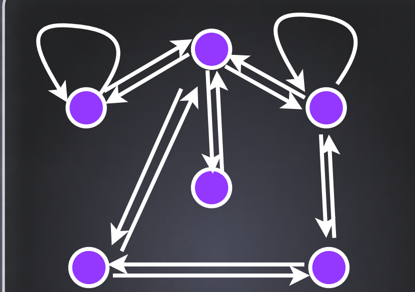




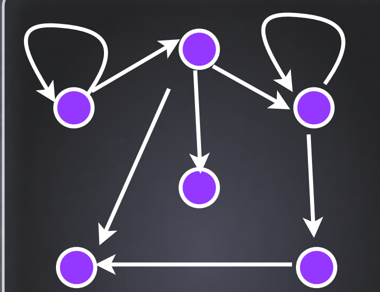
Reflexive:
All self-loops



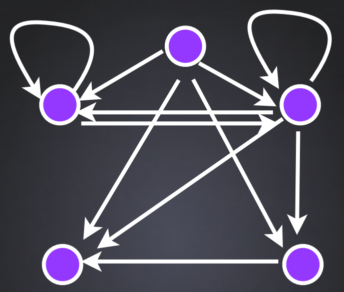
Irreflexive:
No self-loops



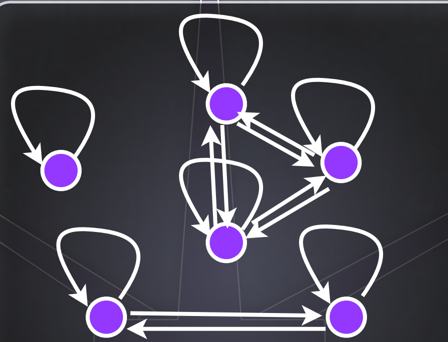
Symmetric:
Only self-loops & bidirectional edges



Anti-symmetric:
No bidirectional edges



Transitive:
Path from a to b implies edge (a,b)



Equivalence:
Cliques, disconnected from each other

Reflexive closure of R: Smallest relation $R' \supseteq R$ s.t. R' is reflexive

Symmetric closure of R: Smallest relation $R' \supseteq R$ s.t. R' is symmetric

Transitive closure of R: Smallest relation $R' \supseteq R$ s.t. R' is transitive

An equivalence relation R is its own reflexive, symmetric and transitive closure