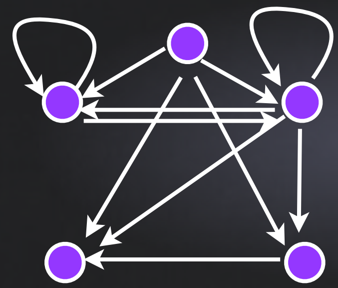
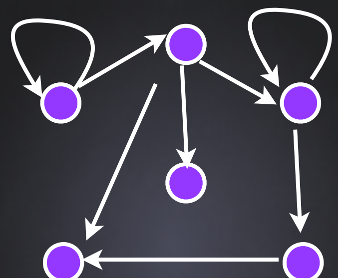
Sets & Relations

Posets

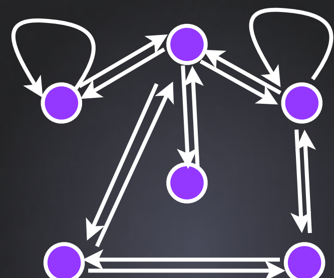
Landscape of Transitive Relations



Transitive:
Path from a to b
implies edge (a,b)



Anti-symmetric:
No bidirectional
edges



Symmetric:
Only self-loops &
bidirectional edges

Acyclic
Cannot follow a sequence of non-self-loop edges and get back to where you started from

$\subseteq$ $\leq$

$<$

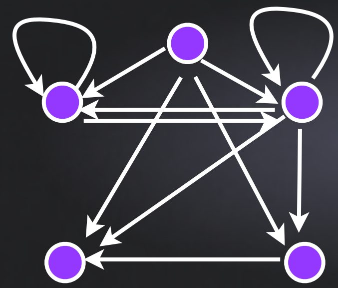
ancestor of

has same last name as

$\equiv$

Anti-Symmetric ⟷ Symmetric

Landscape of Transitive Relations



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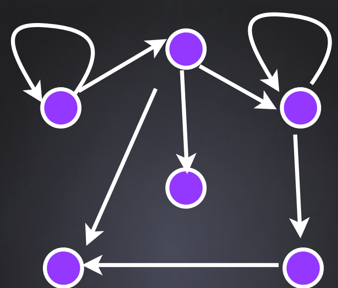

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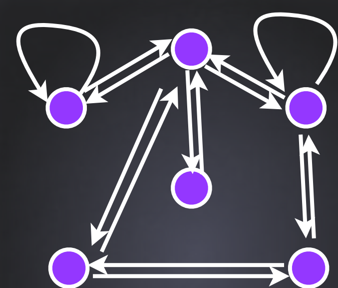

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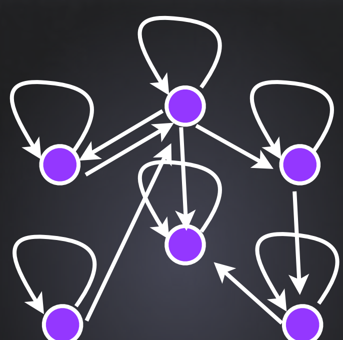
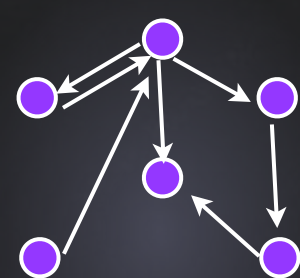
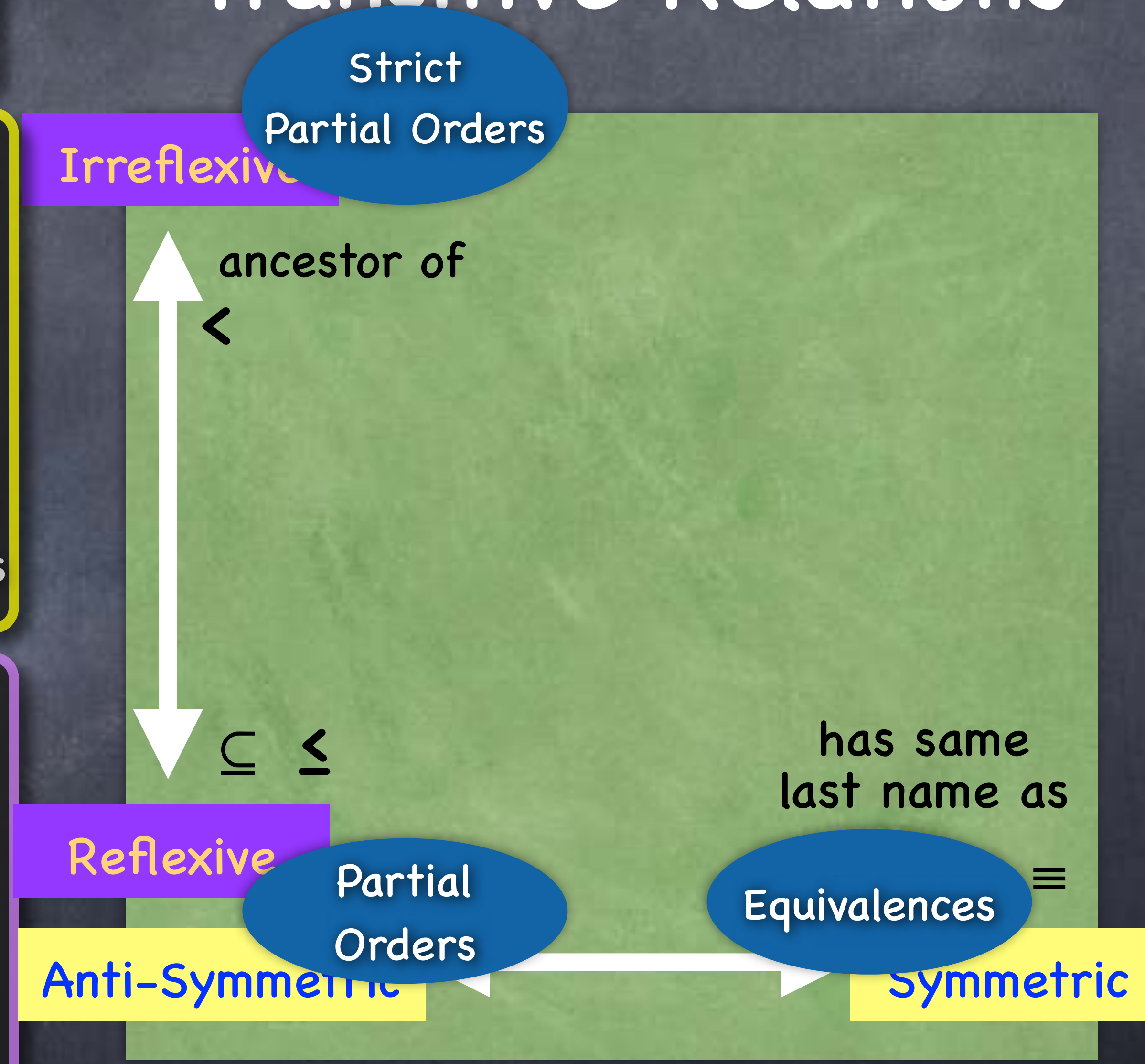


Reflexive:
All self-loops



Irreflexive:
No self-loops



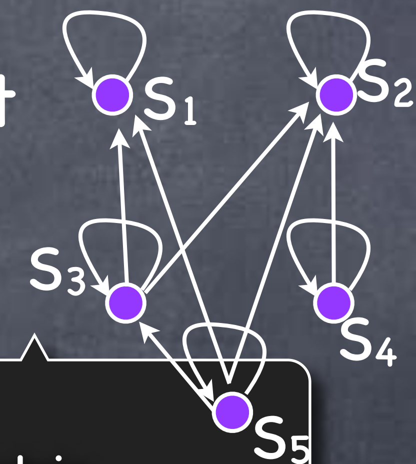
Partial Order

Strict partial order: irreflexive, rather than reflexive

- A transitive, anti-symmetric and reflexive relation
  - e.g. ≤ for integers, divides for integers, ⊆ for sets, "containment" for line-segments
- Equivalently, transitive and acyclic (and ir/reflexive) (a pair of bidirectional edges is a "cycle")
  - "Order" refers to these properties
- "Partial": not every two elements need be "comparable"
  - i.e., {a,b} s.t. neither $a \sqsubseteq b$ nor $b \sqsubseteq a$
    - e.g., neither $A \subseteq B$ nor $B \subseteq A$

Posets

- Partially ordered set (a.k.a Poset)
  - A non-empty set and a partial order over it
  - Denoted like $(S, \preccurlyeq)$
- e.g. $S = \{S_1,S_2,S_3,S_4,S_5\}$ where $S_1=\{0,1,2,3\}$, $S_2=\{1,2,3,4\}$, $S_3=\{1,2,3\}$, $S_4=\{3,4\}$, and $S_5 = \{2\}$. Poset $(S, \subseteq)$

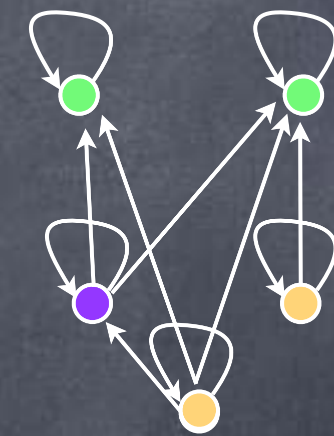


Check:
- Anti-symmetric (no bidirectional edges),
- Transitive,
- Reflexive (all self-loops)

  - More generally, $(S, \subseteq)$ where S is any set of sets
  - Verify: $P\subseteq P$; $P\subseteq Q \wedge Q\subseteq R \rightarrow P\subseteq R$; $P\subseteq Q \wedge Q\subseteq P \rightarrow P=Q$
- e.g. Divisibility poset: $(\mathbb{Z}^+, |\ )$
  - Verify: $a|a$; $a|b \wedge b|c \rightarrow a|c$; $a|b \wedge b|a \rightarrow a=b$

Extremal & Extremum

- Maximal & minimal elements of a poset (S, ≼)
  - x∈S is maximal if ∄y∈S−{x} s.t. x≼y
  - x∈S is minimal if ∄y∈S−{x} s.t. y≼x
    - Need not exist (e.g., in (ℤ,≤)).
    - Need not be unique when it exists (e.g., divisibility poset restricted to integers > 1)



- Claim: Every finite poset has at least one maximal and one minimal element
  - Proof by induction on |S| [Exercise]

Useful in induction proofs about finite posets

- x∈S is the greatest element if ∀y∈S, y≼x
- x∈S is the least element if ∀y∈S, x≼y

Need not exist. Unique when one exists.

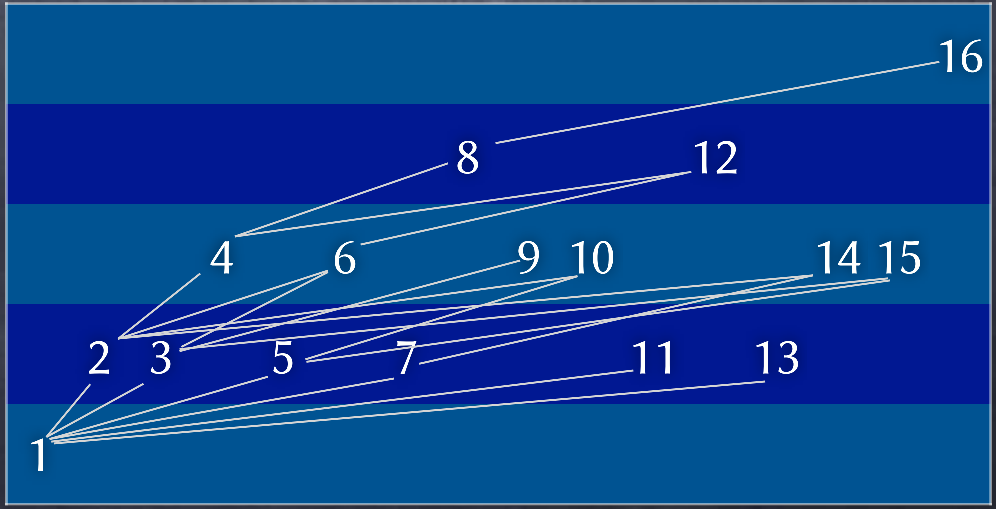
Other Relations from a Poset

- Consider partial order $\preccurlyeq$
- $\prec$ is the reflexive reduction of $\preccurlyeq$ iff $\preccurlyeq$ is the reflexive closure of $\prec$, and $\prec$ itself is irreflexive
  - $a \prec b$ iff $a \neq b$ and $a \preccurlyeq b$
- $\sqsubseteq$ is the transitive reduction of $\preccurlyeq$ iff $\preccurlyeq$ is the transitive closure of $\sqsubseteq$, and $\forall a,b$ ($a \sqsubseteq b \rightarrow \nexists m \notin \{a,b\}$, $a \preccurlyeq m \preccurlyeq b$)
  - Well-defined for finite posets: Define $a \sqsubseteq b$ iff $a \preccurlyeq b$ and $\nexists m \notin \{a,b\}$, $a \preccurlyeq m \preccurlyeq b$. [Prove by induction]
  - Need not exist for infinite sets (e.g., for $(\mathbb{R}, \leq)$, $\sqsubseteq$ defined as above is the equality relation)

Running Example

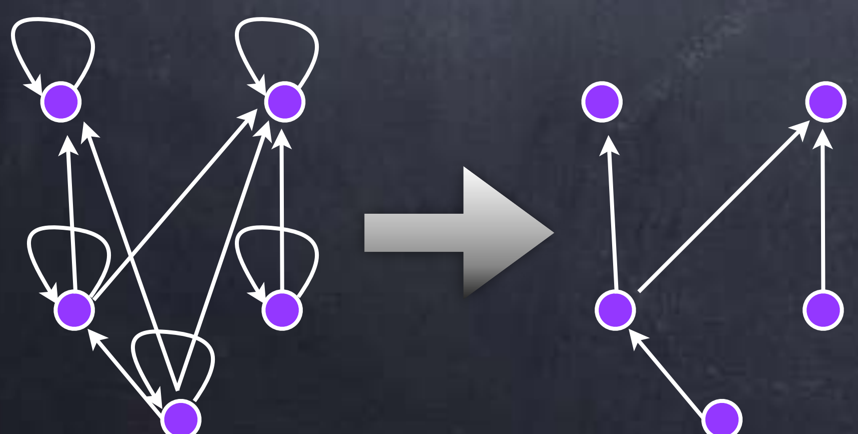
Divisibility poset: ($\mathbb{Z}^+$, |)

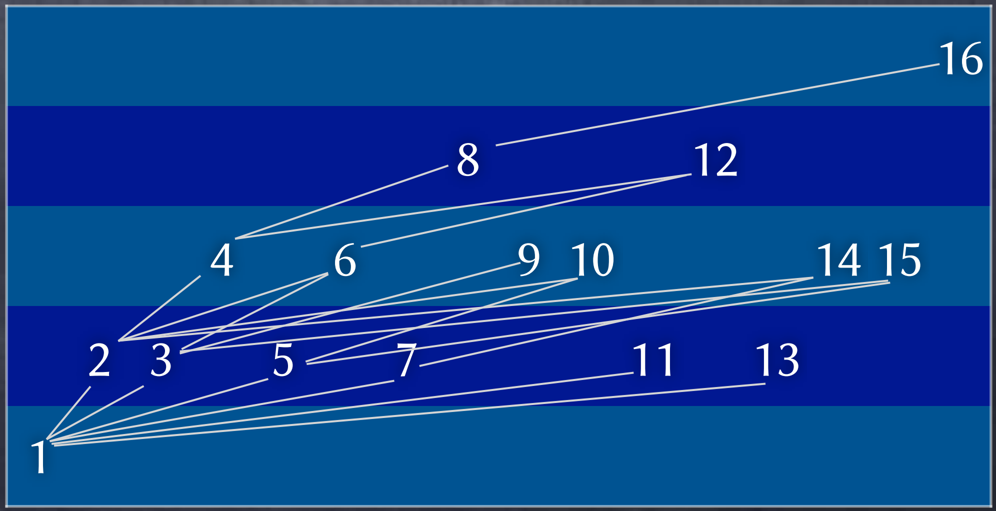
- Consider strict poset ($\mathbb{Z}^+$,⊏), where a ⊏ b iff b/a is prime
- Claim: | is the transitive closure of the reflexive closure of ⊏ [Verify]
- Claim: ⊏ is the transitive reduction of the reflexive reduction of | [Verify]
  - Note: Divisibility poset has a transitive reduction even though it is infinite



Hasse Diagram

- For a poset (S, ≼), the transitive reduction of the reflexive reduction of ≼, if it exists, has all the information about the poset
  - Recall: For finite posets, guaranteed to exist
- Hasse Diagram: the graph of this relation (with arrowheads implicit)





Bounding Elements

- Given a poset $(S, \preccurlyeq)$ and $T \subseteq S$
- Maximal element in T : $x \in T$ s.t. $\forall y \in T$, $x \preccurlyeq y \rightarrow y = x$

  Minimal element in T : $x \in T$ s.t. $\forall y \in T$, $y \preccurlyeq x \rightarrow y = x$
- Greatest element in T : $x \in T$ s.t. $\forall y \in T$ $y \preccurlyeq x$

  Least element in T : $x \in T$ s.t. $\forall y \in T$, $x \preccurlyeq y$
- Upper Bound for T : $x \in S$ s.t. $\forall y \in T$, $y \preccurlyeq x$

  Lower Bound for T : $x \in S$ s.t. $\forall y \in T$, $x \preccurlyeq y$
- Least Upper Bound for T: Least in $\{x |$ x u.b. for T$\}$

  Greatest Lower Bound for T: Greatest in $\{x |$ x l.b. for T$\}$

Need not exist.
Need not be unique when one exists.

Do exist in finite posets

Need not exist.
Unique when one exists.

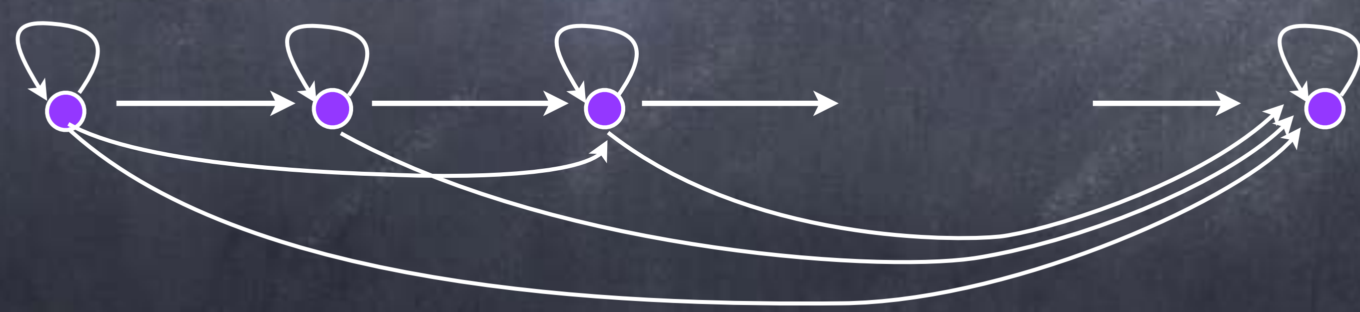
Running Example

Divisibility poset: $(\mathbb{Z}^+, |)$

- Lower bound
  - When is c a lower bound for T={a,b}?
    - $c|a$ and $c|b \Rightarrow$ c is a common divisor for {a,b}
  - Greatest lower bound for {a,b} = gcd(a,b)
- Upper bound
  - d is an upper bound for {a,b} $\Rightarrow a|d,\ b|d \Rightarrow$ d a common multiple for {a,b}
  - Least upper bound for {a,b} = lcm(a,b)

Total/Linear Order

- In some posets every two elements are "comparable": for {a,b}, either a⊑b or b⊑a
  - Can arrange all the elements in a line, with all possible right-pointing edges (plus, all self-loops)



- If finite, has unique maximal and unique minimal elements (left and right ends)

Order Extension

- A poset P'=(S,≤) is an extension of a poset P=(S,≼) if $\forall a,b \in S,\ a \preccurlyeq b \rightarrow a \leq b$
- Any finite poset can be extended to a total ordering (this is called topological sorting)
  - Prove by induction on |S|
    - Induction step: Remove a minimal element, extend to a total ordering, reintroduce the removed element as the minimum in the total ordering.
  - For infinite posets? The "Order Extension Principle" is typically taken as an axiom! (Unless an even stronger axiom called the "Axiom of Choice" is used)

Running Example

Divisibility poset: ($\mathbb{Z}^+$, |)

- The totally ordered set ($\mathbb{Z}^+$, ≤), where ≤ is the standard "less-than-or-equals" relation, is an extension of the divisibility poset
  - Because a|b → a ≤ b
- Consider another totally ordered set ($\mathbb{Z}^+$, ⊑):
  - For any (a,b) ∈ $\mathbb{Z}^+ \times \mathbb{Z}^+$, a⊑b iff:
    - a=1, or
    - a,b both prime or both composite, and a ≤ b, or
    - a prime and b composite
  - ($\mathbb{Z}^+$, ⊑) extends the divisibility poset [Exercise]