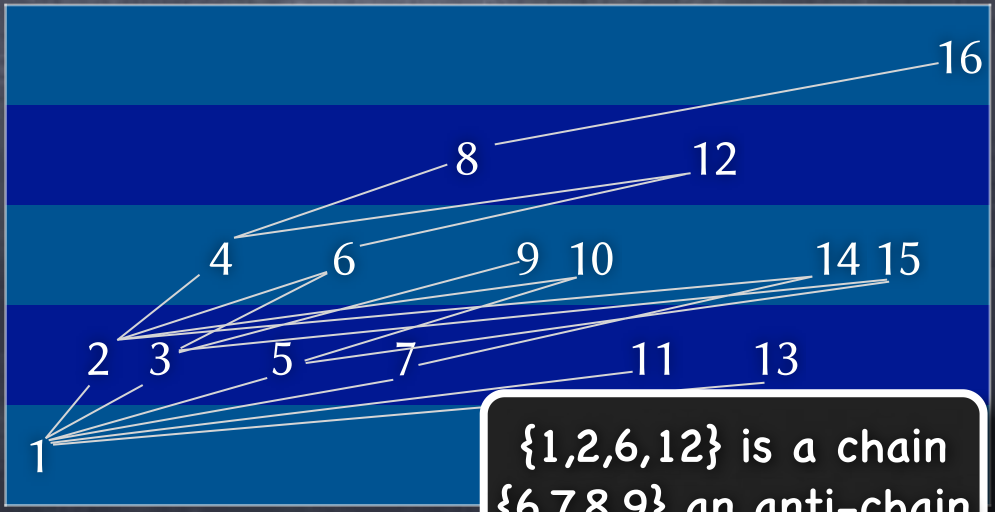
Sets & Relations

Chains and Anti-Chains

Chains & Anti-Chains

In a poset $(S,\preccurlyeq)$

- $C \subseteq S$ is said to be a chain if $\forall a,b \in C$, either $a\preccurlyeq b$ or $b\preccurlyeq a$
- i.e., $(C,\preccurlyeq)$ is a total order
- $A \subseteq S$ is an anti-chain if $\forall a,b\in A$, neither $a\preccurlyeq b$ nor $b\preccurlyeq a$, unless $a=b$
- $(A,\preccurlyeq)$ is same as $(A,=)$
- Subset of a chain is a chain. Similarly for anti-chains.
- A singleton set is both a chain and an anti-chain
- For any chain C and anti-chain A, $|C\cap A| \leq 1$ (Why?)



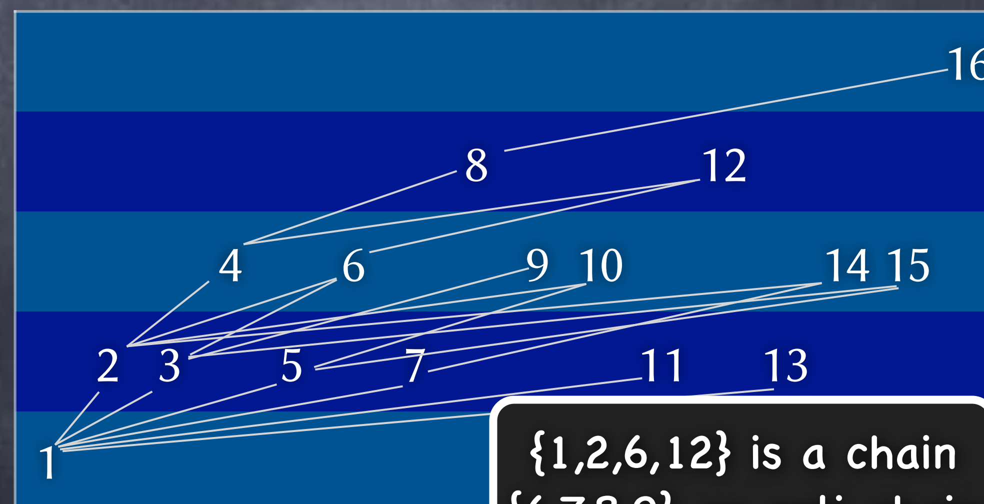
{1,2,6,12} is a chain
{6,7,8,9} an anti-chain
{2,6,7} neither
{6} both

Height in a Poset

- In a poset $(S,\preccurlyeq)$, for any $a \in S$, we define
  height(a) = max size of a chain with a as the maximum
  - Note: every a has {a} as such a chain

Finite if S is finite

- E.g., In $(\mathbb{Z}^+, \mid)$, height(1)=1, height(p)=2 for all primes p.
  For $m = p_1^{d_1} \cdot \ldots \cdot p_t^{d_t}$ (p_i primes), height(m) = $1 + \sum_i d_i$
- Height <u>of</u> the poset $(S,\preccurlyeq)$
  = max { height(a) | $a \in S$}
  = max { |C| | chain C}
  - Size of the largest chain in the poset
  - Possibly ∞ (only if S infinite)



{1,2,6,12} is a chain
{6,7,8,9} an anti-chain
{2,6,7} neither
{6} both

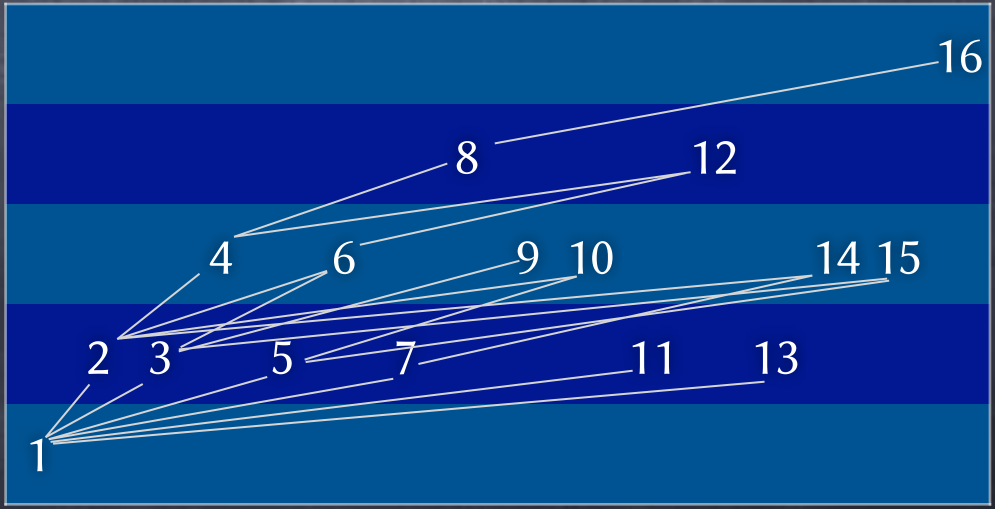
Anti-Chains from Height

- Let A_h = { a | height(a)=h }
- For every finite h, A_h is an anti-chain (possibly empty)
  - Otherwise, $\exists a \neq b$, $a \leqslant b$ with height(a) = height(b) = h.

    height(a) = h $\Rightarrow$ $\exists$chain C s.t. a=max(C) and |C|=h

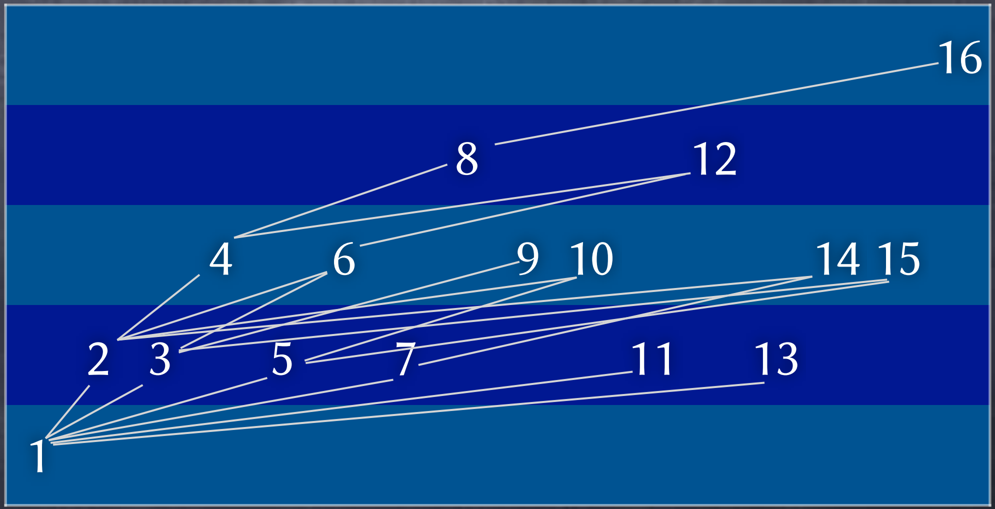
    $\Rightarrow$ $b \notin C$ and $C' = C \cup \{b\}$ is a chain with b=max(C')

How? $\Rightarrow$ height(b) $\geq$ h+1 !



Anti-Chains from Height

- Let A_h = { a | height(a)=h }
- For every finite h, A_h is an anti-chain (possibly empty)
- max { h | $A_h \neq \emptyset$ } = height of the poset = $\max_{C \text{ chain}} |C|$
- In a finite poset, since every element has a finite height, every element appears in some A_h: i.e., A_h's partition S
- Mirsky's Theorem: The least number of anti-chains needed to partition S is exactly the size of a largest chain
  - For chain $C \subseteq S$, need $\geq |C|$ anti-chains to cover C, as $|C \cap A| \leq 1$ for anti-chain A



Partitioning with (Anti)Chains

- Mirsky's Theorem: The least number of anti-chains needed to partition S is exactly the size of a largest chain

Later

- Dilworth's Theorem: The least number of chains needed to partition S is exactly the size of a largest anti-chain