Counting

Permutations & Combinations

Strings

- Given an alphabet (a finite set) B, we can consider strings of length k, made up of characters from the alphabet
  - e.g., B = {a,b,c}, and a length-5 string σ = aacca
  - Formally, a length k string is a function $\sigma : \{1,\ldots,k\} \rightarrow B$

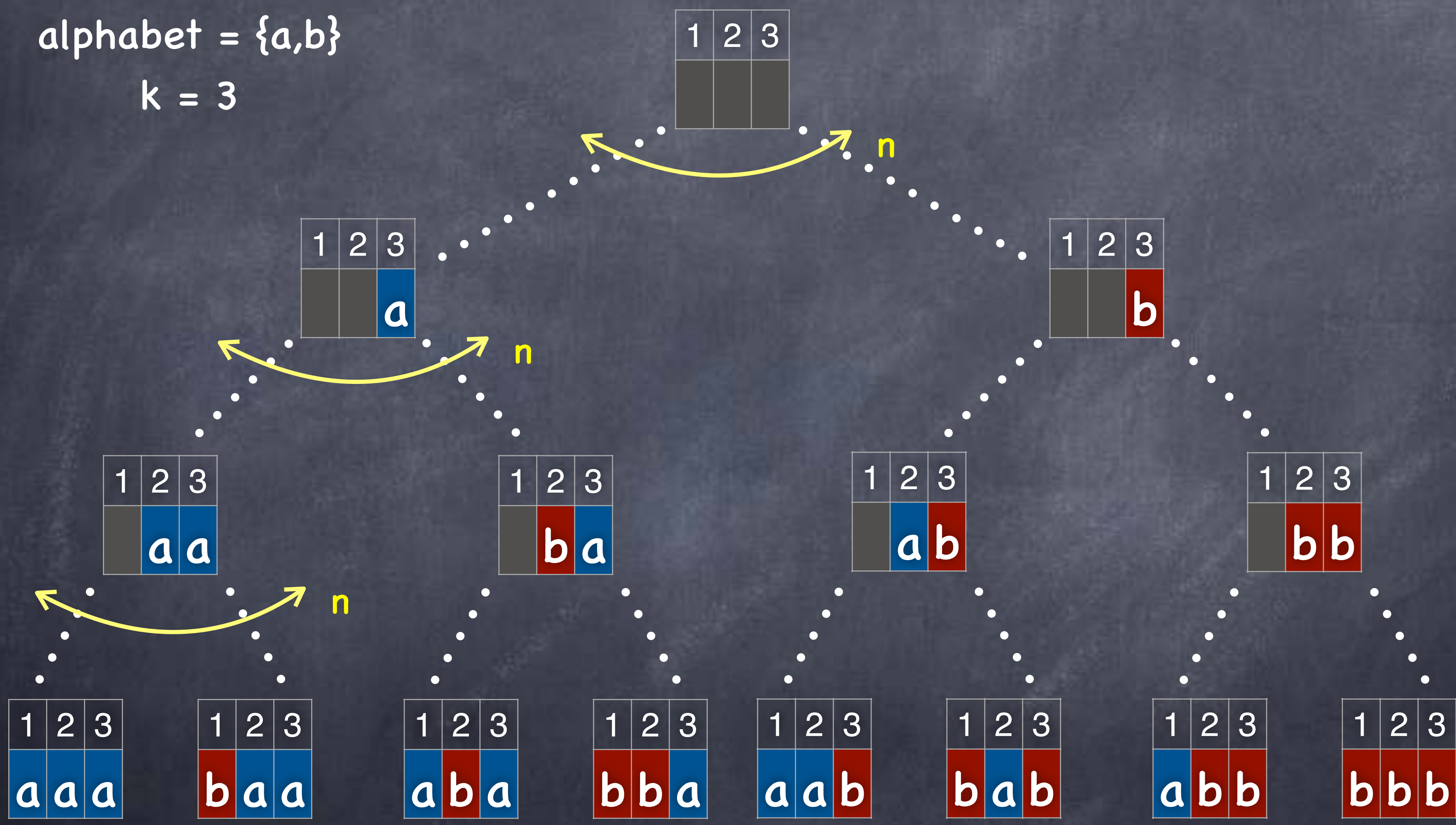
1	2	3	4	5
a	a	c	c	a

- How many length-k strings exist over an alphabet of size n?
  - n^k [Note: Grows exponentially with the length]
  - Proof by induction: Fix arbitrary alphabet size n. Let the number of k-long strings be $\alpha(k)$. Claim $\alpha(k) = n^k$.
    - $\alpha(1) = n$. For k>1, a k-long string consists of a (k-1)-long string followed by a single character. $\alpha(k) = \alpha(k-1)\cdot n$.

Strings

alphabet = {a,b}

k = 3



Binary Strings

- Binary string: A string with alphabet of size 2
  - Typically, alphabet {0,1}
- Number of length-k strings binary strings = 2^k
- A length-k binary string can be used to represent a subset of a set of size k
- Take the alphabet to be [k] $\triangleq$ {1,...,k}
- Subset associated with string σ: $S_\sigma = \{ i \mid \sigma_i = 1 \}$
- Number of subsets of [k] = 2^k

1	2	3	4	5
0	1	0	0	1

$\{2,5\} \subseteq [5]$

Permutations

- Permutations refer to arrangements of a set of symbols as a string, without repetition
  - e.g.,

1	2	3	4	5
c	a	d	e	b

(alphabet = {a,b,c,d,e})

  - A bijection from [n] = {1,...,n} to the alphabet of size n
  - Sometimes we want to consider shorter strings without repeating symbols

1	2	3
c	a	d

One-to-one

- How many length-k strings which do not have repeating symbols exist over an alphabet of size n?
  - $P(n,k) = \begin{cases} 0 & \text{if } k>n \\ n!/(n-k)! & \text{otherwise} \end{cases}$

$$n! = \begin{cases} 1 & \text{if } n=0 \\ n\cdot(n-1)! & \text{if } n>0 \end{cases}$$

Permutations

- How many length-k strings which do not have repeating symbols exist over an alphabet of size n?
  - $P(n,k) = \begin{cases} 0 & \text{if } k>n \\ n!/(n-k)! & \text{otherwise} \end{cases}$
- Proof by induction on n (for all k) [Exercise]
  - Base case, n=1
  - Induction step: Using $P(n,k) = n \cdot P(n-1,k-1)$
    - Alternately, $P(n,k) = P(n,k-1) \cdot (n-k+1)$
- $n!/(n-k)! = \underbrace{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}_{k \text{ times}}$

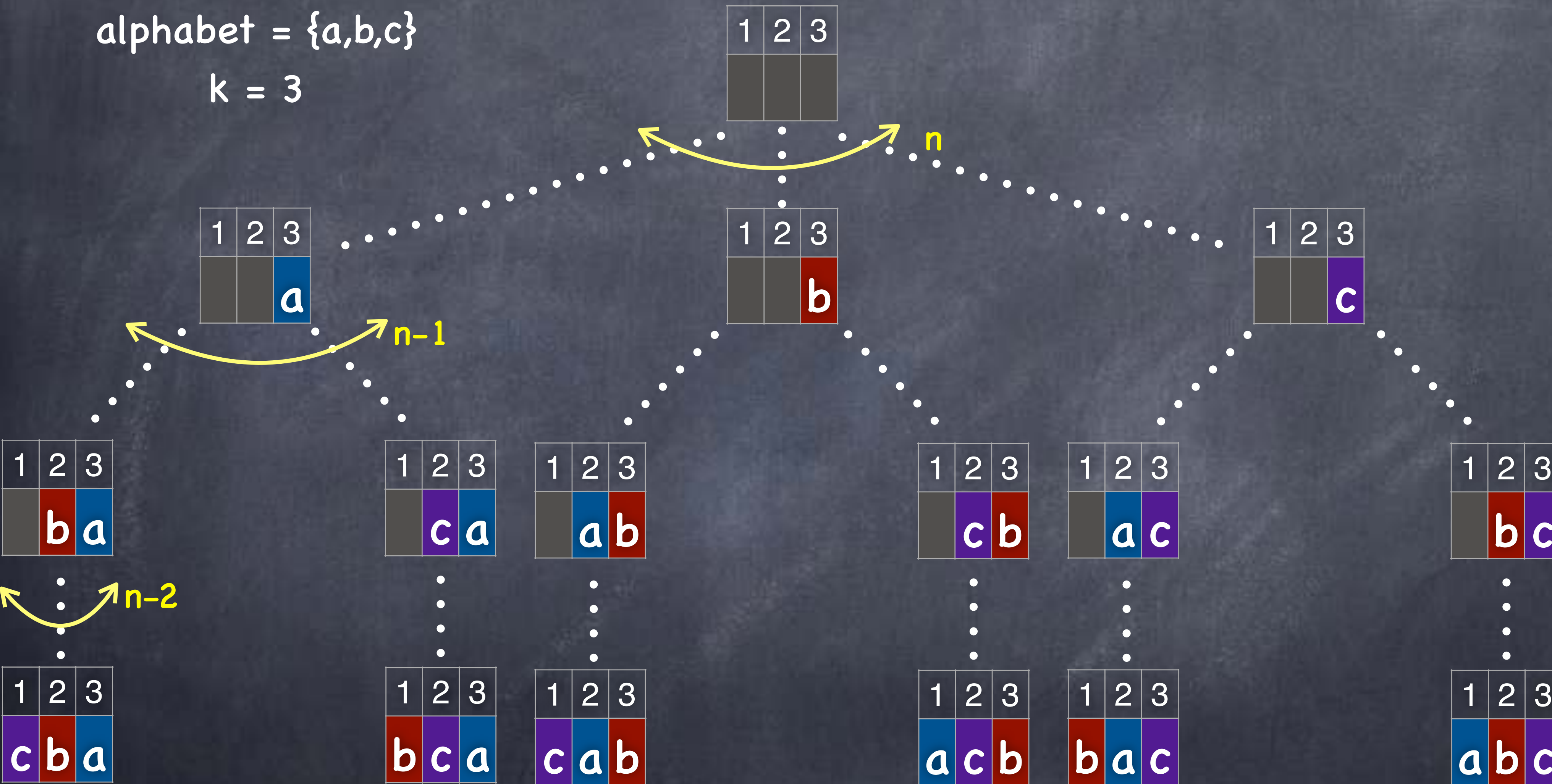
a.k.a. falling factorial, $(n)_k$

- $P(n,n) = n!$

Permutations

alphabet = {a,b,c}

k = 3



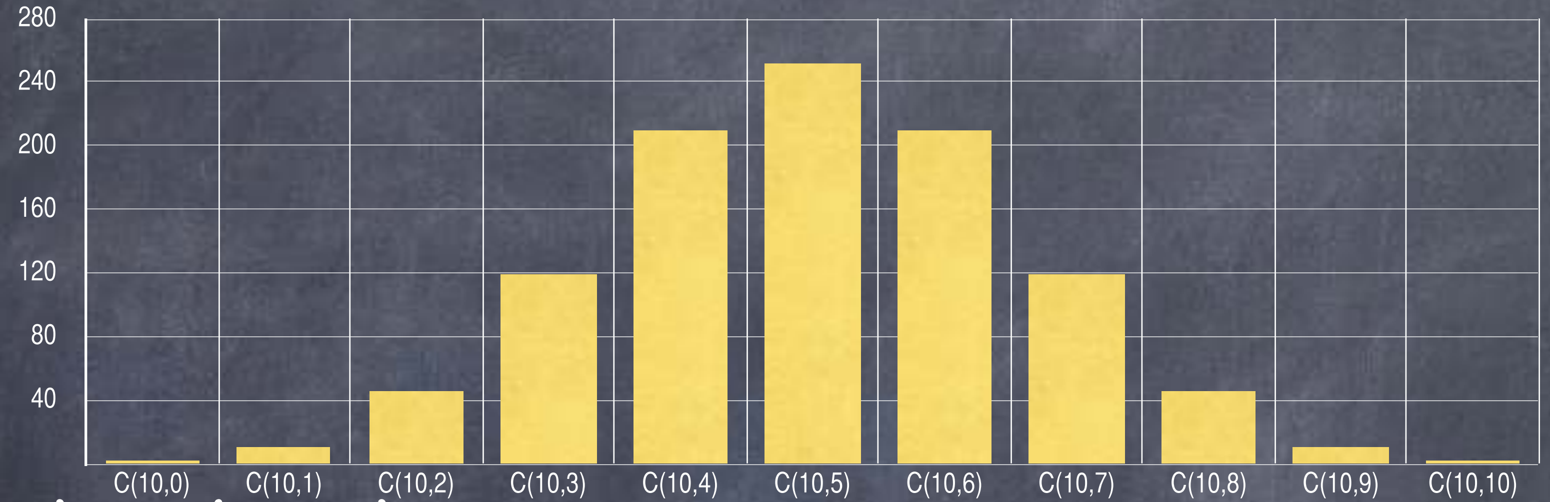
Combinations

- How many subsets of size k does a set of size n have?
- We can represent subsets as strings without repetitions
  - e.g., $\{a,c,d\} \subseteq \{a,b,c,d,e\}$ can be represented as acd
- But the same subset can be represented as multiple strings: adc, cad, ...
  - We know exactly how many ways
  - k! strings using the same k symbols
- # k-symbol subsets of n-symbol alphabet
  = # repetition-free strings of length k, divided by k!
  - $C(n,k) = P(n,k)/k! = n! \,/\, (\,(n-k)! \cdot k!\,)$

Also written $\binom{n}{k}$

C(n,k)

- For $n,k \in \mathbb{N}$, $C(n,k) = n!/(k!(n-k)!)$ if $k \leq n$, and 0 otherwise



- $C(n,k) = C(n,n-k)$
  - Selecting k out of n elements is the same as unselecting n-k out of n elements
- $C(n,0) = C(n,n) = 1$
  - In particular, $C(0,0) = 1$
    (how many subsets of size 0 does $\emptyset$ have?)
- $C(n,0) + C(n,1) + \ldots + C(n,n-1) + C(n,n) = 2^n$

C(n,k)

- $(1+x)^n = \Sigma_{k=0 \text{ to } n}\ C(n,k)\ x^k$
- $(1+x)\cdot(1+x)\cdot(1+x) = (1+x)\cdot(1\cdot 1 + 1\cdot x + x\cdot 1 + x\cdot x)$

  $= 1\cdot 1\cdot 1 + 1\cdot 1\cdot x + 1\cdot x\cdot 1 + 1\cdot x\cdot x$

  $+ x\cdot 1\cdot 1 + x\cdot 1\cdot x + x\cdot x\cdot 1 + x\cdot x\cdot x$

  - Each term is of the form ?·?·? where each ? is 1 or x
- Coefficient of x^k = number of strings with exactly k x's out of the n positions = C(n,k)
- Proof by induction on n:
  coefficient of x^k in $(1+x)\cdot(\ldots + ax^{k-1} + bx^k + \ldots)$ is a+b
  - a = coefficient of x^{k-1} in $(1+x)^{n-1}$ = C(n−1,k−1)
    b = coefficient of x^k in $(1+x)^{n-1}$ = C(n−1,k)
  - C(n,k) = C(n−1,k−1) + C(n−1,k) (where n,k ≥ 1)

C(n,k)

- <u>$C(n,k) = C(n-1,k-1) + C(n-1,k)$</u> (where $n,k \geq 1$)
  - Easy derivation: Let $|S|=n$ and $a \in S$.
    $C(n,k) =$ # k-sized subsets of S containing a
    + # k-sized subsets of S not containing a
- In fact, gives a <u>recursive definition</u> of $C(n,k)$
  - <u>Base case</u> (to define for $k \leq n$): $C(n,0) = C(n,n) = 1$ for all $n \in \mathbb{N}$
  - Or, to define it for all $(n,k) \in \mathbb{N} \times \mathbb{N}$
    <u>Base case:</u> $C(n,0)=1$, for all $n \in \mathbb{N}$, and $C(0,k)=0$ for all $k \in \mathbb{Z}^+$

n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	2	1	0	0	0	0
3	1	3	3	1	0	0	0
4	1	4	6	4	1	0	0
5	1	5	10	10	5	1	0
6	1	6	15	20	15	6	1

Conventions for n=0 or k=0

- # of length-k strings over an alphabet of size n = n^k
  - What if k=0?
  - We define the empty string as a valid string
  - $n^0 = 1$ such string
  - What if n=0? Empty string can be defined over an empty alphabet as well. So, 1 again.
- The empty string has no repeating symbols: $P(n,0) = 1$
  - $P(n,0) = n!/(n-0)!$ still holds
  - $P(0,0) = 1$ holds too since $0! = 1$
- Size-0 subsets of a size-n set? There is just one: $\emptyset$
  - $C(n,0) = 1$. $C(n,0) = n!/(0!\cdot n!)$ still holds
  - $C(0,0) = 1$ (since $\emptyset \subseteq \emptyset$)