# Counting

Balls and Bins

# Balls and Bins

- How many ways can I throw a set of balls into a set of bins?
- Variants based on whether they are considered distinguishable (labelled) or indistinguishable

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- Further variants: "no bin empty", "at most one ball in a bin"

# Balls and Bins

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- Each ball must be thrown into a single bin
  - Throwing: mapping a ball to a bin
  - A function with the set of balls as the domain and the set of bins as the co-domain
- Number of ways of throwing:
  - Number of functions from A to B
    - "Function table": A string of length |A|, over the alphabet B
    - $|B|^{|A|}$ such strings

| $x \in A$ | $f(x) \in B$ |
|---|---|
| 1 | 1 |
| 2 | 0 |
| 3 | 1 |
| 4 | 0 |

# How many Functions?

| | Labelled balls | Unlabelled balls |
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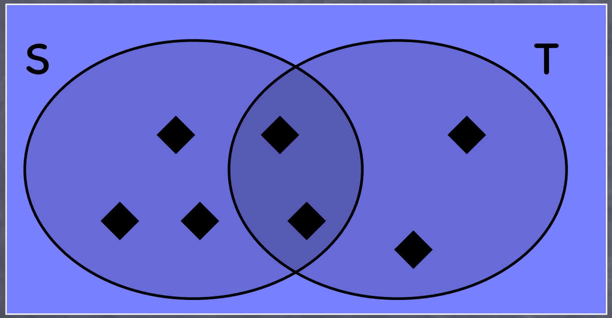
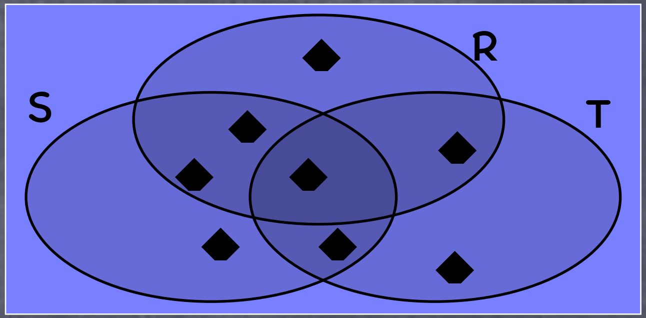
- Balls $\in$ A, bins $\in$ B. Let $|A|=k$, $|B|=n$.
- Unrestricted version:
  - # functions $f: A \rightarrow B = n^k$
- Every bin can hold at most one ball: One-to-one functions
  - # one-to-one functions from A to B = $P(n,k)$
    - Recall Pigeonhole Principle: There is a one-to-one function from A to B only if $|B| \geq |A|$. $P(n,k) = 0$ for $k>n$
  - # bijections from A to B (only if $|A|=|B|$) is $P(n,n) = n!$
- No bin empty: Onto functions
  - # onto functions? A little more complicated.

# Inclusion-Exclusion

- $|S \cup T| = |S| + |T| - |S \cap T|$




- $|R \cup S \cup T| = |R| + |S| + |T| - |R \cap S| - |S \cap T| - |T \cap R| + |R \cap S \cap T|$
- Given n finite sets $T_1, \ldots, T_n$

$$\left| \bigcup_{i \in [n]} T_i \right| = \sum_{J \subseteq [n],\ J \neq \emptyset} (-1)^{|J|+1} \left| \bigcap_{j \in J} T_j \right|$$

- Prove by induction on n [Exercise]
  - $| \cup_{i \in [n+1]} T_i | = | (\cup_{i \in [n]} T_i) \cup T_{n+1} |$

$= |\cup_{i \in [n]} T_i| + |T_{n+1}| - |\cup_{i \in [n]} Q_i|$ where $Q_i = T_i \cap T_{n+1}$ for $i \in [n]$

# Onto Functions

| | Labelled balls | Unlabelled balls |
|---|---|---|
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- How many onto functions from A to B? Say A=[k], B=[n].
  - Let's call it N(k,n)

$n^k - C(n,1)\,(n-1)^k + C(n,2)\,(n-2)^k - \ldots$

- Claim: $N(k,n) = \Sigma_{i=0 \text{ to } n}\ (-1)^i\ C(n,i)\ (n-i)^k$
- Non-onto functions: $\cup_{i\in[n]} T_i$ where $T_i = \{ f{:}A\to B \mid i \notin Im(f) \}$
- Inclusion-exclusion to count $|\cup_{i\in[n]} T_i|$

$|\ \cup_{i\in[n]} T_i\ | = \Sigma_{J\subseteq[n],J\neq\emptyset}\ (-1)^{|J|+1}\ |\cap_{j\in J} T_j|$

  - $|\cap_{j\in J} T_j\ | = (n-t)^k$ where $t=|J|$
    - $f \in T_{i_1} \cap\ldots\cap T_{i_t} \leftrightarrow Im(f) \subseteq [n] - \{i_1,\ldots,i_t\}$
    - $|T_{i_1} \cap \ldots \cap T_{i_t}| = (n-t)^k$
  - Number of $J\subseteq[n]$ s.t. $|J|=t$ is $C(n,t)$
  - $|\cup_{i\in[n]} T_i| = \Sigma_{t\in[n]}\ (-1)^{t+1}\ C(n,t)\ (n-t)^k$
- $N(k,n) = n^k - \Sigma_{t\in[n]}\ (-1)^{t+1}\ C(n,t)\ (n-t)^k = \Sigma_{t=0 \text{ to } n}\ (-1)^t\ C(n,t)\ (n-t)^k$

# Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

<table>
<tr><th></th><th>Labelled balls</th><th>Unlabelled balls</th></tr>
<tr><td>Labelled bins</td><td>Function<table><tr><td>all</td><td>$n^k$</td></tr><tr><td>1-to-1</td><td>P(n,k)</td></tr><tr><td>onto</td><td>N(k,n)</td></tr></table></td><td>Multiset</td></tr>
<tr><td>Unlabelled bins</td><td>Set Partition</td><td>Integer Partition</td></tr>
</table>

# Balls and Bins

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- Throwing k unlabelled balls into n distinguishable bins is the same as assigning integers (number of balls) to each bin
  - But the total number of balls is fixed to k
- A multi-set (a.k.a "bag") is like a set, but allows an element in it to occur one or more times
  - Only multiplicity, not order, matters: e.g., [a,a,b] = [a,b,a]
  - Formally, specified as a multiplicity function: $\mu : B \rightarrow \mathbb{N}$ e.g., $\mu(a)=2$, $\mu(b)=1$, $\mu(x) = 0$ for other x.
  - Size of a multi-set: sum of multiplicities: $\Sigma_{x \in B} \mu(x)$
- Throwing: Making a multi-set of size k, with elements coming from a ground-set of n elements (the n bins)

# Examples

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- Making a multi-set of size k, with elements coming from a ground-set of n elements
  - Place orders for k books from a catalog of n books (may order multiple copies of the same book)
  - Fill a pencil box that can hold k pencils, using n types of pencils
  - Distribute k candies to n kids (kids are distinguishable, candies are not)
  - Solve the equation $x_1 + \ldots + x_n = k$ with $x_i \in \mathbb{N}$
    - Ground-set of size n, $\{a_1,\ldots,a_n\}$. $\mu(a_i)=x_i$.
    - Can think of $x_1,\ldots,x_n$ as the bins, and each ball as a 1

# Stars and Bars

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- How many ways can I throw k (indistinguishable) balls into n (distinguishable) bins?
- Each such combination can be represented using n-1 "bars" interspersed with k "stars"
  - e.g., 3 bins, 7 balls: ★ ★ ★ | ★ ★ ★ | ★
    Or, | | ★ ★ ★ ★ ★ ★ ★ (first two bins are empty)
  - Number of such combinations = ?
    - (n-1)+k places. Choose n-1 places for bars, rest get stars
    - C ( n+k-1, k) ways

# Example

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- How many solutions are there for the equation x+y+z = 11, with x,y,z $\in \mathbb{Z}^+$?
- 3 bins, 11 balls: But no bin should be empty!
- First, throw one ball into each bin
- Now, how many ways to throw the remaining balls into 3 bins?
  - 3 bins, 8 balls
  - 2 bars and 8 stars: e.g., ★ | | ★ ★ ★ ★ ★ ★ ★
  - C(10,2) solutions
    - e.g., above distribution corresponds to x=2, y=1, z=8
- Same as k-n balls, n bins without the no-bin-empty restriction

# Variants

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- Unrestricted use of bins
  - Multi-set of size k, ground-set of size n
  - Stars and Bars: C(n+k-1,n-1)
- No bin empty
  - Multiset of size k, with every multiplicity ≥ 1
    - Multiset of size k-n (with multiplicities ≥ 0)
    - C(k-1,n-1)
- At most one ball in each bin
  - Set of size k
  - C(n,k)

# Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

| | Labelled balls | | | Unlabelled balls | | |
|---|---|---|---|---|---|---|
| Labelled bins | Function | all | $n^k$ | Multiset | all | C(n+k-1,k) |
| | | 1-to-1 | P(n,k) | | 1-to-1 | C(n,k) |
| | | onto | N(k,n) | | onto | C(k-1,n-1) |
| Unlabelled bins | Set Partition | | | Integer Partition | | |

# Balls and Bins

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
| Unlabelled bins | Set Partition | Integer Partition |

- (Labelled) elements of the set A are partitioned into (unlabelled) bins
  - Recall: $\{P_1,\ldots,P_d\}$ is a partition of A if $A = P_1 \cup \ldots \cup P_d$, for all distinct i,j, $P_i \cap P_j = \emptyset$, and no part $P_i$ is empty
- How many partitions does a set A of k elements have?
  - S(k,n): #ways A can be partitioned into exactly n parts
    - This corresponds to the "no bin empty" variant
  - #ways A can be partitioned into at most n parts: $\Sigma_{m\in[n]}$ S(k,m)
  - Total number of partitions, $B_k = \Sigma_{m\in[k]}$ S(k,m)

Stirling number of the 2nd kind

Bell number

# How many Partitions?

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
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- S(k,n): #ways A can be partitioned into exactly n parts
  - Suppose we labeled the parts as 1,...,n
  - Such a partition is simply an onto function from A to [n]
    - N(k,n) ways
  - But in a partition, the parts are not labelled. With labelling, each partition was counted n! times.
  - S(k,n) = N(k,n) / n!

# Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

| | Labelled balls | | | Unlabelled balls | | |
|---|---|---|---|---|---|---|
| Labelled bins | Function | all | $n^k$ | Multiset | all | C(n+k-1,k) |
| | | 1-to-1 | P(n,k) | | 1-to-1 | C(n,k) |
| | | onto | N(k,n) | | onto | C(k-1,n-1) |
| Unlabelled bins | Set Partition | all | $\Sigma_{m\in[n]}$ S(k,m) | Integer Partition | | |
| | | 1-to-1 | 0 or 1 | | | |
| | | onto | S(k,n) | | | |

# Balls and Bins

| | Labelled balls | Unlabelled balls |
|---|---|---|
| Labelled bins | Function | Multiset |
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- Writing k as the sum of n non-negative integers
  - Integer solutions to $x_1 + \ldots + x_n = k$, s.t. $0 \leq x_1 \leq \ldots \leq x_n$
- "No bin empty" variant: $x_i$ are positive integers
  - Number of such solutions called the partition number $p_n(k)$
- Number of solutions for the unrestricted variant: $p_n(k+n)$
  - $x_1 + \ldots + x_n = k$ s.t. $0 \leq x_1 \leq \ldots \leq x_n$
    $\leftrightarrow y_1 + \ldots + y_n = k+n$ s.t. $1 \leq y_1 \leq \ldots \leq y_n$ where $y_i = x_i+1$
- "At most one ball in a bin" variant: 1 if $n \geq k$, 0 otherwise

# Partition Number

| | Labelled balls | Unlabelled balls |
|---|---|---|
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- $p_n(k) = \left| \{ (x_1,\ldots,x_n) \mid x_1+\ldots+x_n=k,\ 1 \le x_1 \le\ldots\le x_n\} \right|$
- $p_0(0) = 1$, if $k>0$ $p_0(k) = 0$, and if $k<n$ $p_n(k) = 0$
- $p_n(k) = p_n(k-n) + p_{n-1}(k-1)$
  - $\left| \{ (x_1,\ldots,x_n) \mid x_1+\ldots+x_n=k,\ 1 < x_1 \le\ldots\le x_n \} \right|$
    $+ \left| \{ (x_1,\ldots,x_n) \mid x_1+\ldots+x_n=k,\ 1 = x_1 \le\ldots\le x_n \} \right|$

| n \ k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 |
| 3 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 |
| 4 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 5 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

# Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

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|---|---|---|---|---|---|---|
| Labelled bins | Function | all | $n^k$ | Multiset | all | C(n+k-1,k) |
| | | 1-to-1 | P(n,k) | | 1-to-1 | C(n,k) |
| | | onto | N(k,n) | | onto | C(k-1,n-1) |
| Unlabelled bins | Set Partition | all | $\Sigma_{m\in[n]} S(k,m)$ | Integer Partition | all | $p_n(k+n)$ |
| | | 1-to-1 | 0 or 1 | | 1-to-1 | 0 or 1 |
| | | onto | S(k,n) | | onto | $p_n(k)$ |