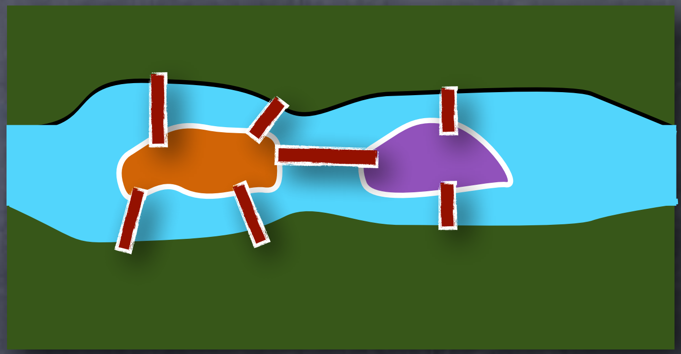
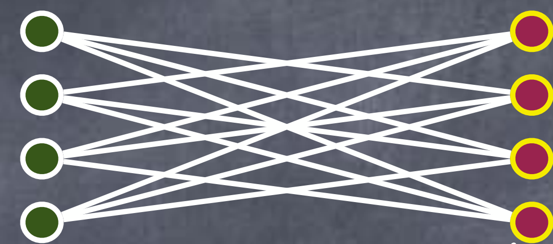
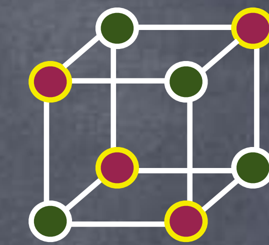
Graphs

Graph Colouring



Graph Colouring





- Recall bi-partite graphs
  - We can "colour" the nodes using 2 colours (which part they are in) so that no edge between nodes of the same colour
- More generally, a **colouring** (using k colours) is proper if there is no edge between nodes of the same colour
  - **k-colouring**: a function $c : V \to \{1,..,k\}$ s.t.
    $$\forall x,y \in V \quad \{x,y\} \in E \to c(x) \neq c(y)$$
  - The least number of colours possible in a proper colouring of G is called the **Chromatic number** of G, $\chi(G)$
    - G has a k-colouring $\leftrightarrow$ $\chi(G) \leq k$
    - G has no k-1-colouring $\leftrightarrow$ $\chi(G) \geq k$

Colouring is Upper-bounding $\chi(G)$

Graph Colouring

- Suppose H is a subgraph of G. Then:
  - G has a k-colouring → H has a k-colouring
    - i.e., $\chi(G) \geq \chi(H)$

Lower-bounding $\chi(G)$

- e.g., G has K_n as a subgraph → $\chi(G) > n-1$ (i.e., $\chi(G) \geq n$)
- e.g., G has C_n for odd n as a subgraph → $\chi(G) > 2$ (coming up)
- Summary: One way to show $k_{lower} \leq \chi(G) \leq k_{upper}$

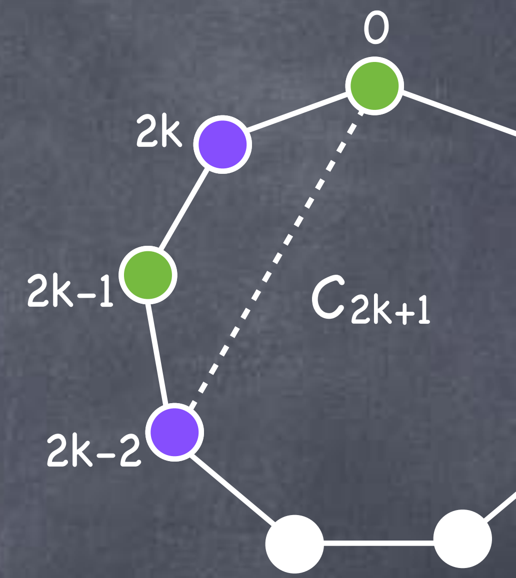
  Show a colouring $c:V \rightarrow \{1,\ldots,k_{upper}\}$

  And show a subgraph H with $k_{lower} \leq \chi(H)$
- Isomorphism preserves χ (exercise)

Graph Colouring

- The origins: map-making
  - "Graph": one node for each country; an edge between countries which share a border
  - Neighbouring countries shouldn't have the same colour. Use as few colours as possible.
- Efficient algorithms known for colouring many special kinds of graphs with as few colours as possible
  - But computing chromatic number in general is believed to be "hard" (it is NP-hard)

Bi-partite Graph

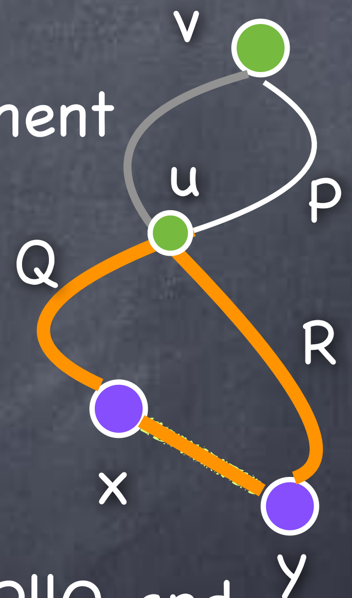


- Claim: for all integers $n \geq 1$, C_{2n+1} is <u>not</u> bi-partite
- Base case: n=1. C_3 has chromatic number 3. ✔
- Induction step: For all integers $k \geq 2$:
  Induction hypothesis: C_{2k-1} is not bi-partite (corresponds to n=k-1)
  To prove: C_{2k+1} is not bi-partite (corresponds to n=k)
  - Will prove contrapositive: C_{2k+1} bi-partite $\rightarrow$ C_{2k-1} bi-partite
  - Suppose a proper 2-colouring $c:\{0,..,2k\} \rightarrow \{1,2\}$ of C_{2k+1}.
  - Then, $c(0) \neq c(2k) \neq c(2k-1) \neq c(2k-2)$. i.e., $c(0)=c(2k-1) \neq c(2k-2)$.
  - Only edge in C_{2k-1} not in C_{2k+1} is $\{0,2k-2\}$.
  - So c respects all edges of C_{2k-1}.
  - So $c':\{0,..,2k-2\} \rightarrow \{1,2\}$ with $c'(u)=c(u)$ a proper colouring of C_{2k-1}.

When G has no odd cycle, this gives a 2-colouring

Bi-partite Graph

- Theorem: G (with |V|>1) is bipartite iff it contains no odd cycle
- To prove: If G not bipartite then it has an odd cycle
  - G (|V|>1) not bipartite ⇒ some such connected component
  - Fix v in this component and partition its nodes as A = { x | dist(x,v) is even }, B = { x | dist(x,v) is odd }
  - Not bipartite ⇒ ∃ edge e={x,y} where x,y∈A or x,y∈B
  - W.l.o.g shortest paths from v to x,y are of the form P||Q and P||R where Q, R are paths from u to x,y and intersect only at u
  - Q and R are both even or both odd length
  - Cycle Q||e||R^{rev} is an odd cycle



Complete Graph

- Suppose G has n nodes. Then, $\chi(G)=n \leftrightarrow$ G is isomorphic to K_n
  - $\leftarrow$: Cannot colour K_n with < n colours (by pigeonhole principle, two nodes with same colour!), and so $\chi(K_n) = n$.
  - $\rightarrow$: We will prove the contrapositive: i.e., that if G with n nodes is not isomorphic to K_n, then $\chi(G)\neq n$.
    - Suppose $G=(V,E)$ is not isomorphic to $K_{|V|}$
      $\Rightarrow \exists$ distinct $u,v \in V$ s.t. $\{u,v\} \notin E$
      $\Rightarrow$ A proper colouring which assigns the same colour to both u and v, and $|V|-2$ other colours to other nodes
      $\Rightarrow \chi(G) \leq |V|-1$

Cliques and Independent Sets

- Clique number $\omega(G)$: Largest k s.t. G has a subgraph isomorphic to K_k
  - $\chi(G) \geq \omega(G)$
- Independence number $\alpha(G)$: Largest k s.t. G has a set of k nodes with no edges among them
  - Nodes of each colour corresponds to an independent set — so at most $\alpha(G)$ nodes
  - Consider a colouring of G with $\chi(G)$ colours.
  - $n = \sum_c$ #nodes with colour c $\leq \chi(G)\cdot\alpha(G)$
  - $\chi(G) \geq n/\alpha(G)$
- $\chi(G) \leq \Delta(G)+1$

Proof describes a recursive algorithm for colouring with $\Delta(G)+1$ colours

olouring and Degree

r all graphs, $\chi(G) \leq \Delta(G)+1$

Fact: among connected graphs, equality holds only for K_n and C_{2n+1}

- Proof by induction on the number of nodes, n
- Base case: n=1.
  There is only one such graph, for which $\Delta(G)=0$, $\chi(G)=1$
- Induction step: For all integers $k \geq 1$:
  Induction hypothesis: for all G=(V,E) with $|V|=k$, $\chi(G) \leq \Delta(G)+1$
  To prove: for all graphs G=(V,E) with $|V|=k+1$, $\chi(G) \leq \Delta(G)+1$.
  - Let G=(V,E) be an arbitrary graph with $|V|=k+1$.

    Important!
  - Let G′=(V′,E′) be obtained from G by removing some $v \in V$ (i.e., $V' = V - \{v\}$) and all edges incident on it
  - $|V'|=k$. So $\chi(G') \leq \Delta(G')+1 \leq \Delta(G)+1$. Colour G′ with $\Delta(G)+1$ colours.
  - $\deg(v) \leq \Delta(G)$. So colour v with a colour in $\{1,..,\Delta(G)+1\}$ that does not appear in its neighbourhood. Valid colouring.
    So $\chi(G) \leq \Delta(G)+1$.

Graphs in action

Graph Colouring in Action

- Many problems can be modeled as a graph colouring problem
- Resource scheduling: allocate "resources" (e.g. time slots, radio frequencies) to "demands" (exams, radio stations) so that there are no "conflicts." Use as few resources as possible.
  - Create a "conflict graph": Demands are the nodes; connect them by an edge if they have a conflict (same student, inhabited area with signal overlap)
  - Colour the graph with as few colours as possible
  - Allocate one resource per colour. Then, no two demands satisfied by the same resource have a conflict