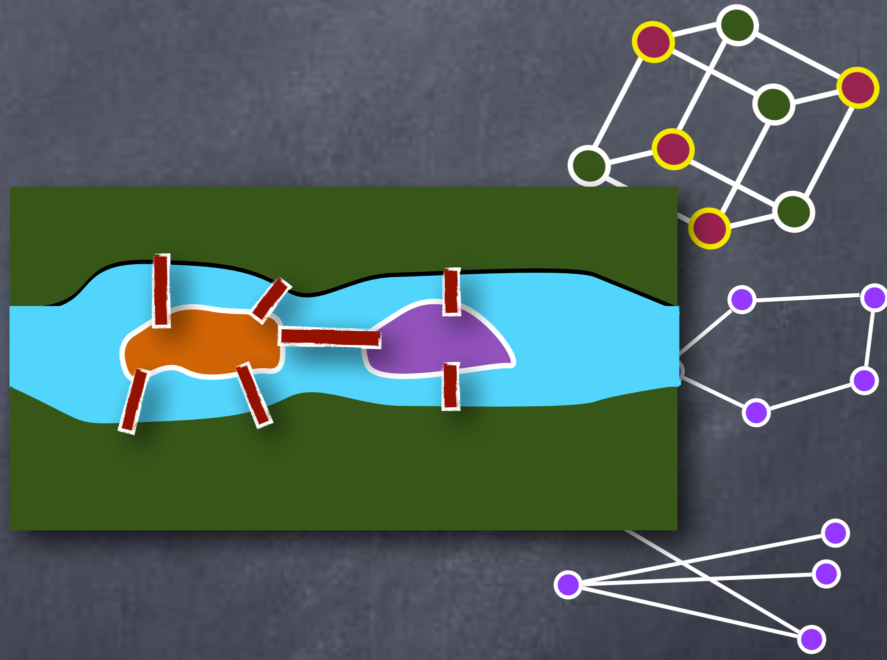
Graphs

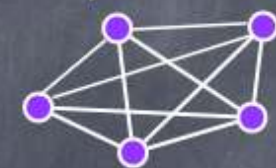
More Examples

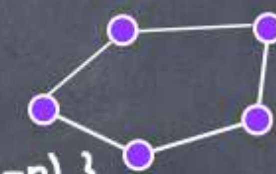


More Examples

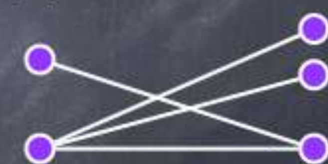
Examples

- Complete graph K_n : n nodes, with all possible edges between them
  - $E = \{ \{a,b\} \mid a,b \in V, a \neq b \}$
  - # edges, $|E| = n(n-1)/2$





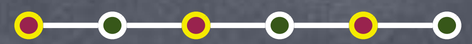
- Cycle C_n : $V = \{ v_1,\ldots,v_n \}$, $E = \{ \{v_i,v_j\} \mid j=i+1 \text{ or } (i=1 \text{ and } j=n) \}$
- Bipartite graph : $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ (i.e., a partition), and no edge between two nodes in the same "part": $E \subseteq \{ \{a,b\} \mid a \in V_1, b \in V_2 \}$
  - e.g., C_n where n is even



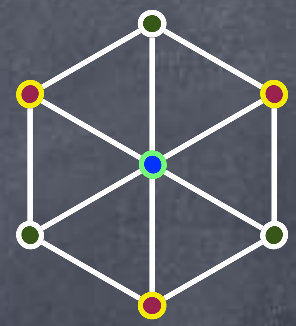
- Complete bipartite graph $K_{n1,n2}$: Bipartite graph, with $|V_1|=n_1$, $|V_2|=n_2$ and $E = \{ \{a,b\} \mid a \in V_1, b \in V_2 \}$
  - # edges, $|E| = n_1 \cdot n_2$

- Later: Hypercube, Trees

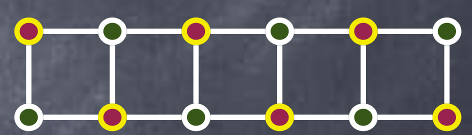
More Examples



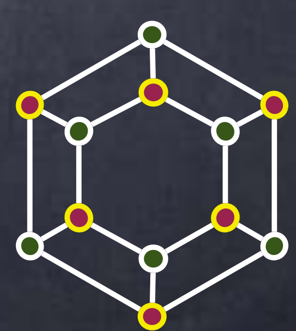
- Path graph P_n :
  $V = \{1,\ldots,n\}$ and $E = \{ \{i,i+1\} \mid i \in [n-1] \}$



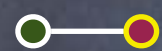
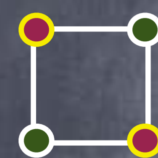
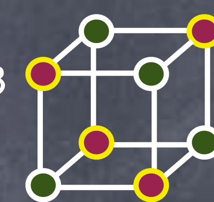
- Wheel graph W_n ($n \geq 3$):
  $V = \{hub\} \cup \mathbb{Z}_n$ and
  $E = \{ \{hub, x\} \mid x \in \mathbb{Z}_n\} \cup \{ \{x,x+1\} \mid x \in \mathbb{Z}_n\}$

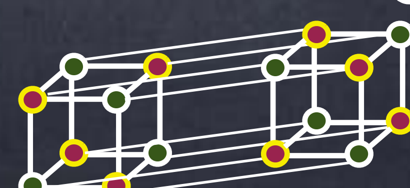


- Ladder graph L_n: $V = \{0,1\} \times \{1,\ldots,n\}$,
  $E = \{ \{(0,i),(1,i)\} \mid i \in [n] \} \cup \{ \{(b,i),(b,i+1) \mid b \in \{0,1\}, i \in [n-1] \}$
  - Circular Ladder graph CL_n:
    2 additional edges $\{(b,n),(b,1)\}$



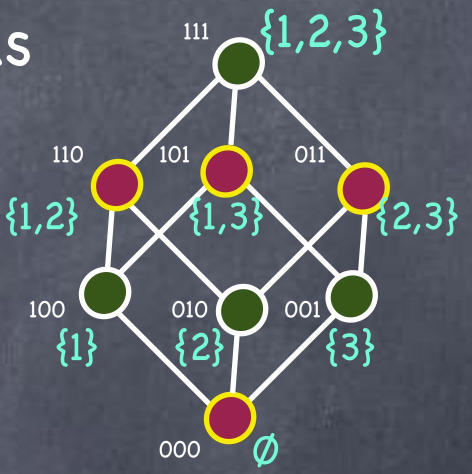
Hypercubes

- The hypercube graph Q_n
  - Nodes: all n-bit strings. e.g., {000, 001, 010, 011, 100, 101, 110, 111}
  - Edges: x and y connected iff they differ in exactly one position
    - i.e., x & y neighbours if toggling a single bit changes x to y
  - e.g.: Q_0  Q_1  Q_2  Q_3 
- 2^n nodes, but "diameter" (longest shortest path) is only n
- Q_n is an n-regular bi-partite graph
  - The two parts: nodes labeled with strings which have <u>even parity</u> (even# 1s) and those labeled with strings of <u>odd parity</u> (odd# 1s)
- Q_{n-1} is a subgraph of Q_n

"$0Q_3$"  "$1Q_3$"

Kneser Graph

- Instead of bit strings, nodes in Q_n can be taken as subsets of [n], with edges present between sets which differ in a single element



- Graph $\overline{KG}_{n,k}$ has nodes at the k^{th} level in Q_n: i.e., subsets of size k
  - Edge between subsets which intersect
  - A clique in $\overline{KG}_{n,k}$: a set of subsets which intersect pairwise.
    - E.g., $\{ \{n\} \cup S \mid S \subseteq [n-1], |S|=k-1 \}$, has $C(n-1,k-1)$ nodes
    - **Erdős-Ko-Rado Theorem:** If $k \le n/2$, then no larger cliques
- Kneser Graph $KG_{n,k}$: Same as above, but with edges between subsets which are disjoint

Graph Operations

- Complement: Interchange edges and non-edges
  - Given $G = (V,E)$, $\overline{G} = (V,\overline{E})$
- Union, Intersection, Difference, Symmetric difference:
  - $G_1 = (V,E_1)$, $G_2 = (V,E_2)$, G_1 op G_2 = $(V, E_1$ op $E_2)$
- Union and intersection can also be defined for $G_1 = (V_1,E_1)$, $G_2 = (V_2,E_2)$ as G_1 op G_2 = $(V_1$ op V_2, E_1 op $E_2)$
  - Disjoint union: Union when $V_1 \cap V_2 = \emptyset$
- Powering
  - Given $G = (V,E)$, the square of G, $G^2 = (V,E')$ where $E' = E \cup \{ \{x,y\} \mid \exists w \; \{x,w\},\{w,y\} \in E \}$
  - More generally, G^k has an edge $\{x,y\}$ iff G has a path of length $t \in [k]$ between x and y

Graph Operations

- Cross product
  - If $G_1 = (V_1,E_1)$, $G_2 = (V_2,E_2)$, then $G_1 \times G_2 = (V_1 \times V_2, E)$, where $\{(u_1,u_2),(v_1,v_2)\} \in E$ iff $\{u_1,v_1\} \in E_1$ and $\{u_2,v_2\} \in E_2$
  - e.g., $G \times K_2$ is a bipartite graph
- Box product
  - $G_1 \square G_2 = (V_1 \times V_2, E)$, where $\{(u_1,u_2),(v_1,v_2)\} \in E$ iff $(\{u_1,v_1\} \in E_1$ and $u_2=v_2)$ or $(u_1=v_1$ and $\{u_2,v_2\} \in E_2)$
  - e.g., $Q_m \square Q_n = Q_{m+n}$
  - e.g., Hamming graph (yields hypercubes for $q=2$)
    - $H_{n,q}$ is $K_q \square \cdots \square K_q$ (n copies)
    - Vertex set: $[q] \times \cdots \times [q]$ (n copies). Edge between $(u_1,\ldots,u_n)$ and $(v_1,\ldots,v_n)$ s.t. $u_i=v_i$ for all but one coordinate.