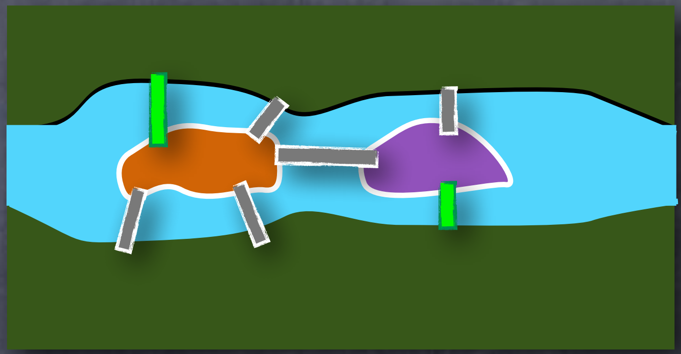
Graphs

Matching

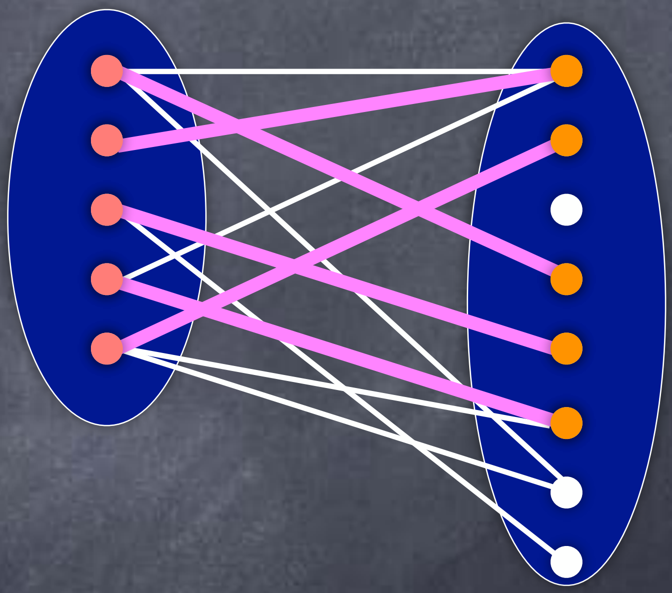


Matchings

- A **matching** in a graph G=(V,E) is a set of edges which do not share any vertex
  - i.e., a set $M \subseteq E$ s.t. $\forall e_1, e_2 \in M$, $e_1 \neq e_2 \rightarrow e_1 \cap e_2 = \emptyset$
  - Every node gets "matched" with at most one other node
- Trivial matchings: $\emptyset$ is a valid matching. For any $e \in E$, $\{e\}$ is a valid matching, too.
- A perfect matching: All nodes are matched by M.
  - i.e., a matching M s.t. $\forall v \in V$, $\exists\ e \in M$ s.t. $v \in e$
  - May or may not exist
- Algorithmic task: given a graph find a largest (maximum) matching
  - Efficient algorithms do exist (we will not cover them here)

Matchings in Bipartite Graphs

- Denote by G=(X,Y,E) a bipartite graph (X∪Y,E) where X,Y≠∅, X∩Y=∅ and, ∀e∈E, |e∩X|=|e∩Y|=1



- Given bipartite G=(X,Y,E), a complete matching from X to Y is a matching M s.t. |M|=|X|
  - Exists only when |X| ≤ |Y| because |M| ≤ min (|X|,|Y|)
- If |X|=|Y|, a complete matching from X to Y is also a complete matching from Y to X
  - And is a perfect matching

Graphs in action

Graph Matching in Action

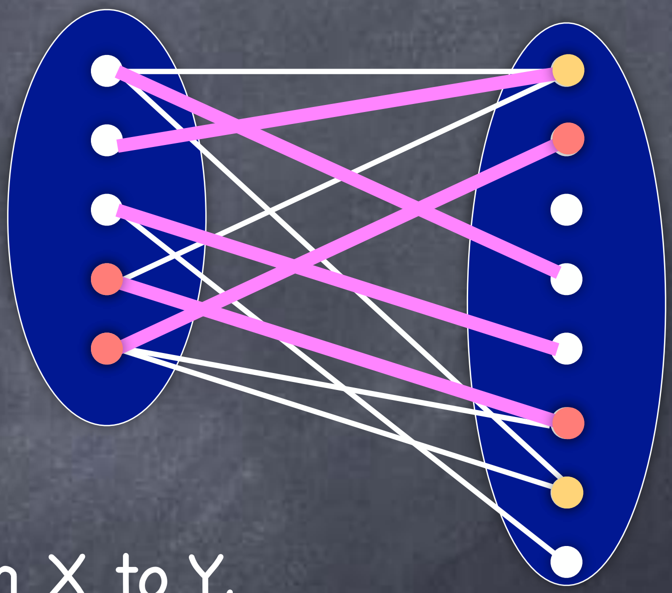
- Matching in bipartite graphs
  - Assigning tasks to workers: a {worker, task} edge if the worker is qualified for the task. A worker should be assigned only one task, and each task needs only one worker.
    - Maximum matching: Getting most tasks assigned to workers
  - Advertisements and slots (e.g., on webpages): each advertiser specifies which slots they prefer; the goal is to maximise the number of slots filled
  - Additional issues: weights (maximum weight matching), costs (e.g., minimum cost perfect matching), "online matching"

Shrinking Neighbourhood

- Given a graph $G = (V,E)$, and $v \in V$, we define v's **neighbourhood:**
  - $\Gamma(\{v\}) \triangleq \{ u \mid \{u,v\} \in E \}$
- More generally, neighbourhood of a set $S \subseteq V$:
  - $\Gamma(S) \triangleq \bigcup_{v \in S} \Gamma(\{v\})$
- In a bipartite graph, $G=(X,Y,E)$, consider $S \subseteq X$
  - $\Gamma(S) \subseteq Y$
  - We shall say S is shrinking if $|\Gamma(S)| < |S|$
  - More generally, for $B \subseteq Y$, S shrinking in B if $|\Gamma(S) \cap B| < |S|$
    - i.e., the set of neighbours of S in B is smaller than S

Hall's Theorem

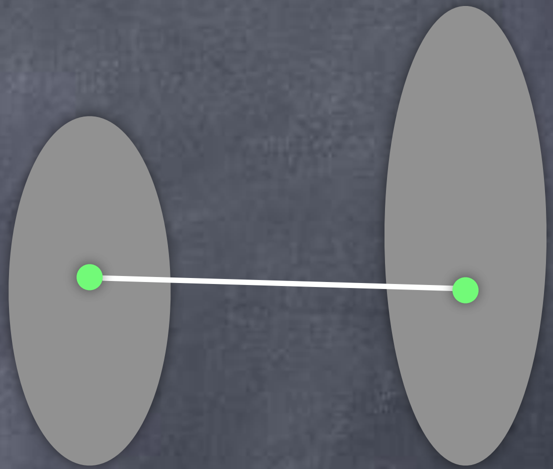
- Bipartite graph G=(X,Y,E) has a <u>**complete matching from X to Y**</u> **iff** <u>**no subset of X is shrinking**</u>
  - i.e., "no shrinking subset" is a necessary and sufficient condition for a complete matching to exist



- Easy direction: Necessary
  - i.e., If there is a complete matching from X to Y, then $\forall S \subseteq X$, S is not shrinking in Y [Why?]
- Proof of sufficiency: Coming up

Hall's Theorem

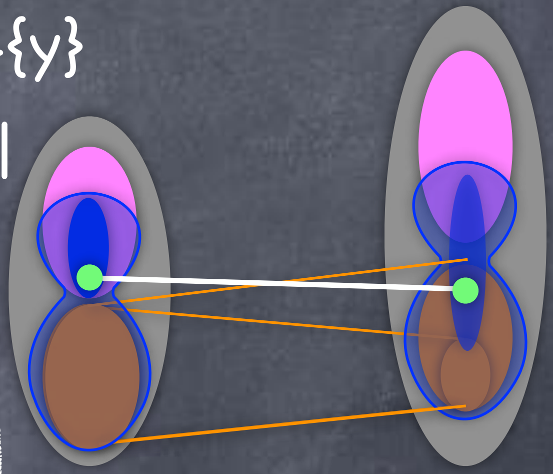
- **Claim**: No shrinking $S \subseteq X \rightarrow \exists$ a complete matching from X into Y
- Proof by strong induction on $|X|$.
- Base case, $|X|=1$: ✓ (How?)
- Induction step: Suppose claim holds for graphs with $|X| \leq k$.



  - Given graph (X,Y,E) with $|X|=k+1$, s.t. $\forall U \subseteq X$, $|\Gamma(U)| \geq |U|$
  - Pick an arbitrary $x \in X$, and an arbitrary neighbour y of x (since $\{x\}$ is not shrinking, x has a neighbour).
  - Case 1: There is a complete matching from $X-\{x\}$ to $Y-\{y\}$.
    Then, X has a complete matching into Y ✓
  - Case 2: No complete matching from $X-\{x\}$ to $Y-\{y\}$.

Hall's Theorem

- Given graph (X,Y,E) with $|X|=k+1$, s.t. $\forall U \subseteq X$, $|\Gamma(U)| \geq |U|$
- Case 2: No complete matching from $X-\{x\}$ to $Y-\{y\}$.
- By ind. hyp., $\exists$ $S \subseteq X-\{x\}$ s.t. S is shrinking in $Y-\{y\}$
  - S shrinking in $Y-\{y\}$ but not in Y. So, $|\Gamma(S)|=|S|$
- Claim: $\exists$ a complete matching from S into $\Gamma(S)$
  - $|S| \leq k$, and no subset of S is shrinking. So by ind. hyp. $\exists$ a complete matching of S into Y. This must be into $\Gamma(S)$
- Claim: $\exists$ a complete matching from X-S into $Y-\Gamma(S)$
  - $|X-S| \leq k$. By ind. hyp., enough to show $\forall T \subseteq X-S$, $|\Gamma(T)-\Gamma(S)| \geq |T|$
  - Consider $U=T \cup S$. $|\Gamma(U)| \geq |U| = |T|+|S|$.
  - Then $|\Gamma(T)-\Gamma(S)| = |\Gamma(U)-\Gamma(S)| = |\Gamma(U)|-|\Gamma(S)| \geq |T|$
- Hence $\exists$ a complete matching from X into Y ✓



Hall's Theorem

- **Claim**: No shrinking $S \subseteq X \rightarrow \exists$ a complete matching from X into Y
- Proof by strong induction on $|X|$.
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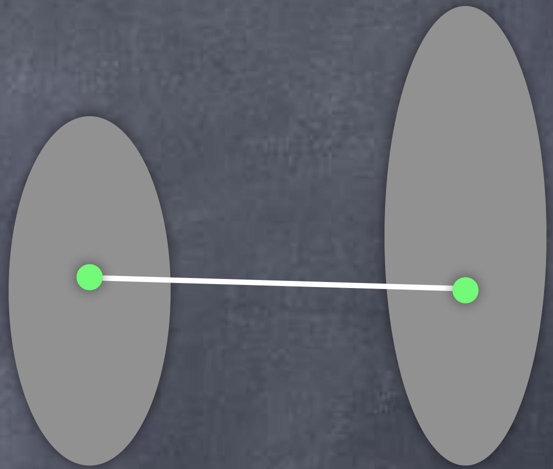

  - Given graph (X,Y,E) with $|X|=k+1$, s.t. $\forall U \subseteq X$, $|\Gamma(U)| \geq |U|$
  - Pick an arbitrary $x \in X$, and an arbitrary neighbour y of x (since {x} is not shrinking, x has a neighbour).
  - Case 1: There is a complete matching from X–{x} to Y–{y}. Then, X has a complete matching into Y ✓
  - Case 2: No complete matching from X–{x} to Y–{y}. ✓

Hall's Theorem
Example Application

- Claim: The edge set of any bipartite graph in which all the nodes have the same degree d can be partitioned into d matchings
  - Note that such a graph G=(X,Y,E) would have $|X|=|Y|=|E|/d$.
- Proof by induction on d.
- For d=1, the graph is a matching. Suppose holds for $d \leq k$.
- Given a bipartite graph G=(X,Y,E) of degree d=k+1. Enough to find one perfect matching M in G.
  - After removing it, will be left with a bipartite graph with degree k for all nodes, and then can use ind. hyp.
- Find a perfect matching: Enough to show that no $S \subseteq X$ is shrinking
- $d|S|$ = #edges incident on S ≤ #edges incident on $\Gamma(S) = d|\Gamma(S)|$

  $\Rightarrow |\Gamma(S)| \geq |S|$ ✓