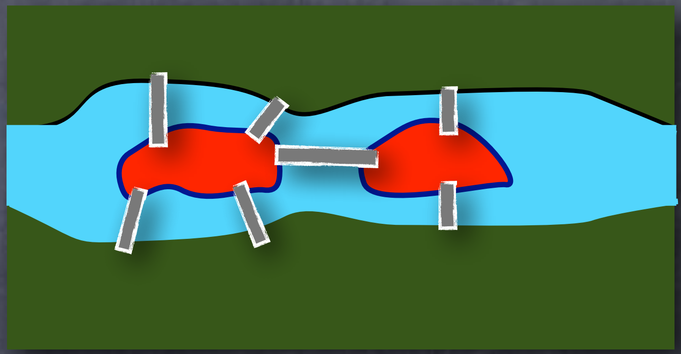
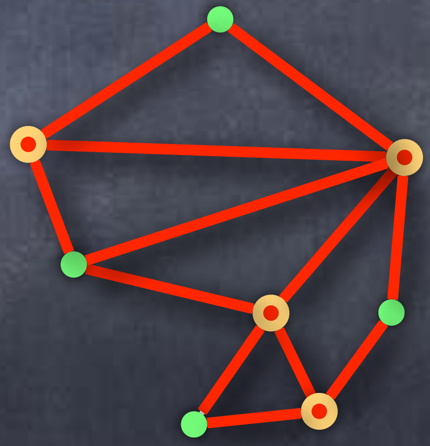
Graphs

Vertex Cover



Vertex Cover

- A vertex cover of a graph G=(V,E) is a set C of vertices such that every edge is covered by (incident on) at least one vertex in C
  - i.e., $C \subseteq V$ is a vertex cover if $\forall\, e \in E,\ e \cap C \neq \emptyset$



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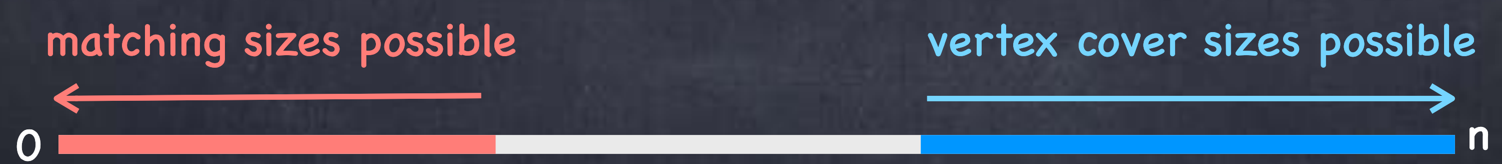
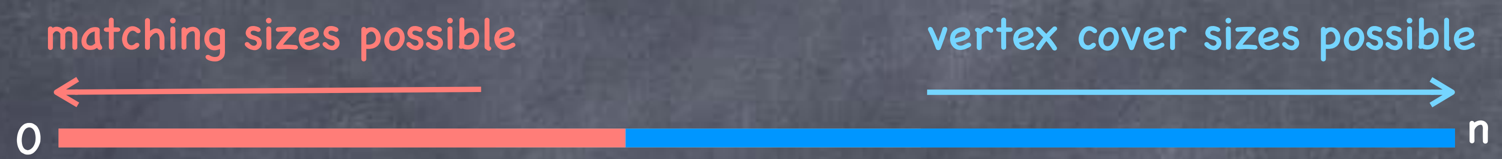
- Trivial vertex covers: V is a vertex cover. So is V-{v}, for any $v \in V$
- Algorithmic task: Find a small vertex cover of a given graph
  - "Hard" (i.e., NP-hard) to find the size of smallest vertex cover
- Two useful results connecting the minimum vertex cover problem to the maximum matching problem (which is not a hard problem)
  - In bipartite graphs, the size of a smallest vertex cover equals the size of a largest matching
  - In general graphs, they are within a factor of 2 of each other

Vertex Cover and Matchings

- A vertex cover of a graph G=(V,E) is a set C of vertices such that every edge is covered by (incident on) at least one vertex in C
  - i.e., $C \subseteq V$ is a vertex cover if $\forall\ e \in E,\ e \cap C \neq \emptyset$
- A matching in a graph G=(V,E) is a set of edges which do not share any vertex
  - i.e., a set $M \subseteq E$ s.t. $\forall e_1, e_2 \in M,\ e_1 \neq e_2 \rightarrow e_1 \cap e_2 = \emptyset$
- In any graph, $\forall$ vertex cover C, $\forall$ matching M, **$|C| \geq |M|$**
  - Because any vertex can cover at most one edge in M



Vertex Cover in Bipartite Graphs



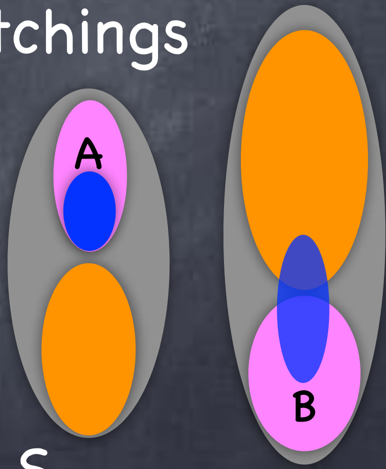
- **Kőnig's Theorem: In a bipartite graph, the size of the smallest vertex cover equals the size of the largest matching**

- Enough to prove that in a bipartite graph G=(X,Y,E), given a smallest vertex cover C, there is a matching M with $|M| \geq |C|$

- Alternately, given a maximum matching M, show a vertex cover C with $|C| \leq |M|$

Vertex Cover in Bipartite Graphs

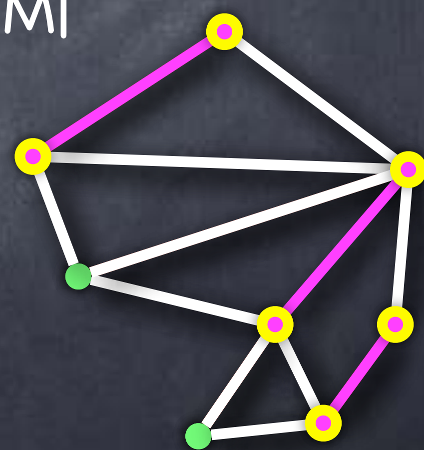
- **Kőnig's Theorem: In a bipartite graph, the size of the smallest vertex cover equals the size of the largest matching**
- Enough to prove that in a bipartite graph G=(X,Y,E), given a smallest vertex cover C, there is a matching M with $|M| \geq |C|$
- Let $A=C \cap X$ and $B=C \cap Y$. Enough to show ∃ complete matchings (1) from A to Y-B and (2) from B to X-A
  - By Hall's theorem, enough to show $\nexists S \subseteq A$ shrinking in Y-B (and similarly $\nexists S \subseteq B$ shrinking in X-A)
    - Suppose $S \subseteq A$ shrinking in Y-B. Consider $C \cup \Gamma(S) - S$
      - Still a vertex cover
        [edges covered by S are covered by $\Gamma(S)$]
      - And strictly smaller than C!
        [$|C \cup \Gamma(S) - S| = |C| + |\Gamma(S) - B| - |S| < |C|$.]



Vertex Cover in General Graphs

- Recall that finding (the size of) a smallest Vertex Cover is hard, but finding a maximum matching isn't
  - Even easier to find a maximal matching: M is a maximal matching if no edge $e \in E-M$ such that $M \cup \{e\}$ is also a matching
    - Repeat until no more edges: pick an arbitrary edge, and delete all edges touching it
- If M a maximal matching, $\exists$ a vertex cover of size $2|M|$
  - Include both end points of each edge in M (i.e., $C = \cup_{e \in M} e$)
  - M is maximal $\Rightarrow \nexists$ e with both its nodes not in C

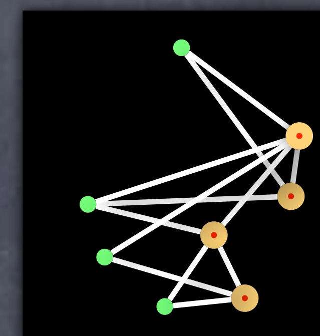
    $\Rightarrow$ C is a vertex cover



- If C is a smallest vertex cover and M a maximal matching, $|M| \leq |C| \leq 2|M|$. Hence, can efficiently approximate the size of the smallest vertex cover within a factor of 2.

Vertex Cover and Independent Set

- In a graph $G=(V,E)$, $I \subseteq V$ is said to be an independent set if there are no edges in the subgraph induced by I
  - i.e., $\forall e \in E, \quad e \not\subseteq I$



- I is an independent set iff $\bar{I}$ is a vertex cover
  - $e \not\subseteq I \longleftrightarrow e \cap \bar{I} \neq \emptyset$
  - I i.s. $\Leftrightarrow \underline{\forall e \in E \;\; e \not\subseteq I} \Leftrightarrow \underline{\forall e \in E \;\; e \cap \bar{I} \neq \emptyset} \Leftrightarrow \bar{I}$ v.c.
- Hence size of smallest v.c. = n - size of largest i.s.