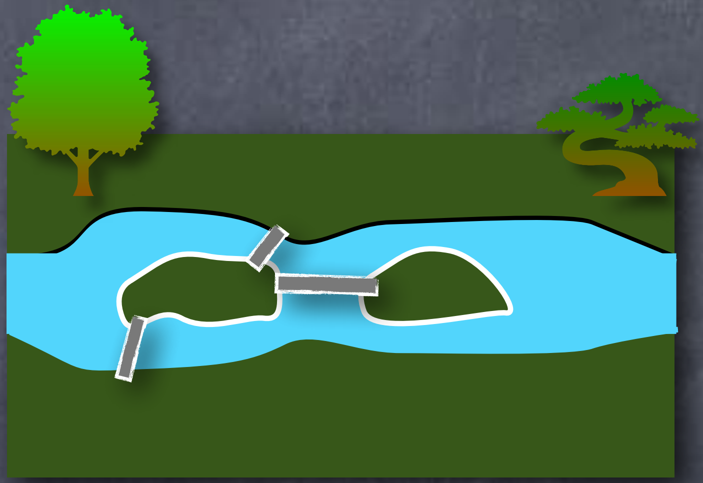
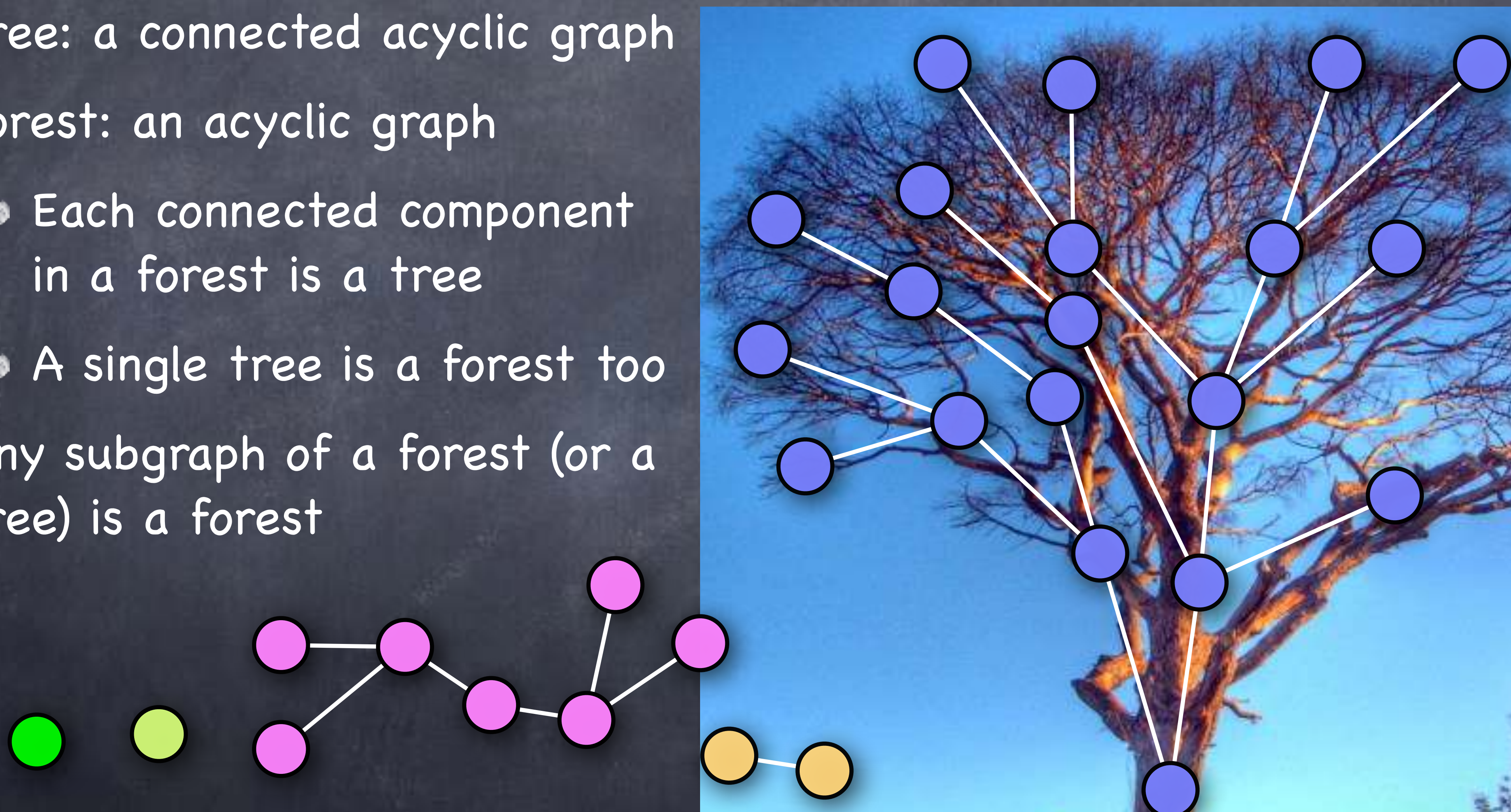
Graphs

Trees

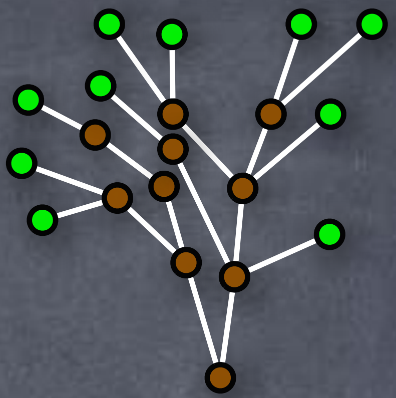


Trees and Forests

- Tree: a connected acyclic graph
- Forest: an acyclic graph
  - Each connected component in a forest is a tree
  - A single tree is a forest too
- Any subgraph of a forest (or a tree) is a forest

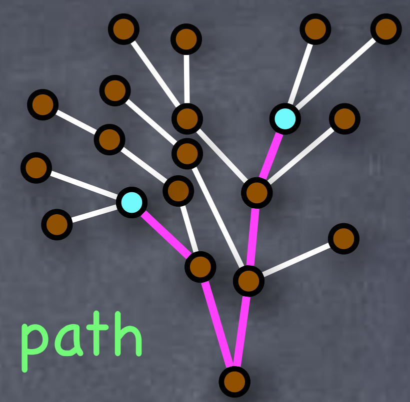


Leafs



- A leaf is a node which has degree 1
- Every tree with at least 2 nodes has at least 2 leaves
  - Consider a maximal path P = $v_0,...,v_k$ [exists in any finite graph]
  - k>0 [else v_0 is an isolated vertex, and the graph is not connected]
  - If v_0 is not a leaf, it has a neighbour v_i for i>1. But then $v_0,...,v_i$ form a cycle! So v_0 is a leaf. Similarly, v_k is a leaf.
- If G is a tree with at least 2 nodes, deleting a leaf w (and the one edge incident on it) results in a tree G'
  - G' is connected, because all u-v paths (i.e., paths from u to v) in G are retained in G' for u,v≠w

Induction on Trees (By Deleting Leafs)



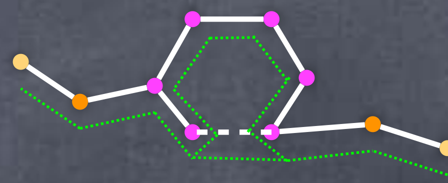
- In a tree, for all nodes u,v, there is exactly one u-v path
- Proof by induction on the number of nodes
- Base case: 1 node. Only one path from v to itself (of length 0) ✓
- Suppose the claim holds for trees with k nodes, for some k≥1.
- Given a tree G with k+1 nodes, delete a leaf w to get a tree G′
  - (Check: There is a leaf, and deleting it gives a tree)
  - For u,v≠w: any u-v path in G is present in G′ (w cannot occur in the middle of a path). So, by ind. hyp. exactly one u-v path.
  - For u≠w, v=w: Any u-w path in G is of the form u-x path followed by w, where x is w's only neighbour. But exactly one u-x path. So exactly one u-w path.
  - Also, only one w-w path.
  - So for all u,v, exactly one u-v path in G ✓

Number of Edges

- In a tree (V,E), |E| = |V|-1
- Proof by induction on |V|
- Base case: |V| = 1. Only one such tree, and it has |E|=0.
- Induction step: for all k ≥ 1
  Hypothesis: for every tree (V,E) with |V|=k, |E|=|V|-1
  To prove: for every tree (V,E) with |V|=k+1, |E|=|V|-1
  - Suppose G=(V,E) is a tree with |V| = k+1. Consider G′=(V′,E′) be the tree obtained by deleting a leaf.
  - By induction hypothesis, |E′| = |V′|-1 = k-1. But |E| = |E′|+1 because exactly one edge was deleted. So |E| = k. ✓

Number of Edges

- In a tree (V,E), $|E| = |V|-1$
- If a graph (V,E) is connected, and $|E|=|V|-1$, then it must be a tree
  - If there is a cycle, can delete any edge in the cycle, and still get a connected graph.



  - Repeat until no more cycles. This is a tree with $|E| < |V|-1$!
- Adding a new edge to a tree makes it cyclic, with a single cycle
- In a forest (V,E), the number of connected components, $c=|V|-|E|$
  - Components be (V_i,E_i). Note that $|V| = \Sigma_i |V_i|$ and $|E| = \Sigma_i |E_i|$
    $|E| = \Sigma_{i=1 \text{ to } c} |E_i| = \Sigma_{i=1 \text{ to } c} (|V_i|-1) = (\Sigma_{i=1 \text{ to } c} |V_i|) - c = |V|-c$
- Deleting a degree d node from a tree makes it a forest with d connected components