Graphs
Dilworth's Theorem
Min-Max Results

Recall:

- In a graph, \( \text{size of any matching} \leq \text{size of any vertex cover} \)
  - In bipartite graphs, equality achieved! Kőnig’s theorem

- In a poset, \( \text{size of any chain} \leq \text{size of any anti-chain decomp} \)
  - Equality is achieved! Mirsky’s theorem

Today:

- In a poset, \( \text{size of any anti-chain} \leq \text{size of any chain decomp} \)
  - Equality is achieved! Dilworth’s theorem

Each chain can have at most one element of an anti-chain.
Dilworth’s Theorem

Dilworth’s Theorem: The least number of chains needed to partition $S$ is exactly the size of a largest anti-chain.

Easy direction: size of any anti-chain $\leq$ size of any chain decomposition.

To prove: There is an anti-chain at least as large as a chain decomposition.

Consider a poset $(S, \preceq)$, with $|S|=n$.

Construct a bipartite graph $G$ s.t.

- A vertex cover of size $\leq t$ in $G \Rightarrow$ antichain of size $\geq n-t$.
- A matching of size $\geq t$ in $G \Rightarrow$ partition $S$ into $\leq n-t$ chains.

Kőnig’s theorem: there is a vertex cover and matching of the same size, say $t$, in $G$.

Hence an antichain at least as large as a chain decomposition.
Dilworth's Theorem

Let $G = (S \times \{0\}, S \times \{1\}, E)$, where $E = \{ \{(u,0),(v,1)\} \mid u \leq v, u \neq v \}$

Given vertex cover $C$, let $B = \{ u \mid \exists b \in \{0,1\}, (u,b) \in C \}$. Let $A = S - B$.

$|B| \leq |C| \Rightarrow |A| \geq |S| - |C|$

Also, $A$ is an anti-chain

[ If $u,v \in A$, and $u \leq v$, then $(u,0)$ and $(v,1) \not\in C$, and edge $\{(u,0),(v,1)\} \in E$! ]
Given a matching $M$, define a graph $F = (S, E^*)$, where $E^* = \{ \{u, v\} \mid \{(u, 0), (v, 1)\} \in M \}$.

$F$ is a forest, with each connected component being a path.

- In $F$, any $u$ will have degree $\leq 2$ [one from $(u, 0)$, one from $(u, 1)$]
- $F$ has no cycles [Cycle $v_0, v_1, \ldots, v_k \Rightarrow v_0 \leq v_1 \leq \ldots \leq v_0 !$]

Each such path in $F$ forms a chain in the poset.

Number of chains = number of connected components

$= |S| - |E^*| = |S| - |M|$
Dilworth’s Theorem

Dilworth’s Theorem: The least number of chains needed to partition S is exactly the size of a largest anti-chain

Easy direction: size of any anti-chain ≤ size of any chain decomposition

To prove: There is an anti-chain at least as large as a chain decomposition

Consider a poset \((S, \preceq)\), with \(|S|=n\)

Construct a bipartite graph \(G\) s.t.

- a vertex cover of size \(\leq t\) in \(G\) ⇒ antichain of size \(\geq n-t\)
- a matching of size \(\geq t\) in \(G\) ⇒ partition \(S\) into \(\leq n-t\) chains

Kőnig’s theorem: there is a vertex cover and matching of the same size, say \(t\), in \(G\)

Hence an antichain at least as large as a chain decomposition
Mirsky’s theorem and Dilworth’s theorem can be seen as statements about the comparison graph of the poset.

Given a poset \((S, \preceq)\), its comparison graph is \(G = (S, E)\) where \(E = \{ \{u, v\} \mid u \preceq v, u \neq v \}\).

If \(G\) is a comparison graph, any induced subgraph of \(G\) is also a comparison graph.

A chain corresponds to a clique in \(G\), and an anti-chain corresponds to an independent set (i.e., a clique in \(\overline{G}\)).

An anti-chain decomposition corresponds to a colouring of \(G\), and a chain decomposition corresponds to a colouring of \(\overline{G}\).

Mirsky’s theorem: If \(G\) is a comparison graph, \(\chi(G) = \omega(G)\)

Dilworth’s theorem: If \(G\) is a comparison graph, \(\chi(\overline{G}) = \omega(\overline{G})\)
Comparison Graph

- Mirsky’s theorem: If $G$ is a comparison graph, $\chi(G) = \omega(G)$
- Dilworth’s theorem: If $G$ is a comparison graph, $\chi(\overline{G}) = \omega(\overline{G})$

- If $G$ is a comparison graph, any induced subgraph of $G$ is also a comparison graph

- **Perfect Graph:** $G$ is a perfect graph if for every induced subgraph $G'$ of $G$, $\chi(G') = \omega(G')$

- Mirsky: A comparison graph is perfect
- Dilworth: The complement of a comparison graph is perfect

- **Fact [Perfect Graph Theorem]:** $G$ is perfect iff $\overline{G}$ is perfect

  - **Note:** Given Perfect Graph Theorem, Mirsky $\Leftrightarrow$ Dilworth