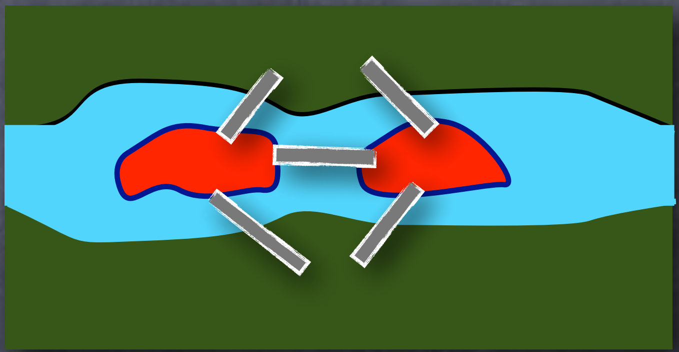
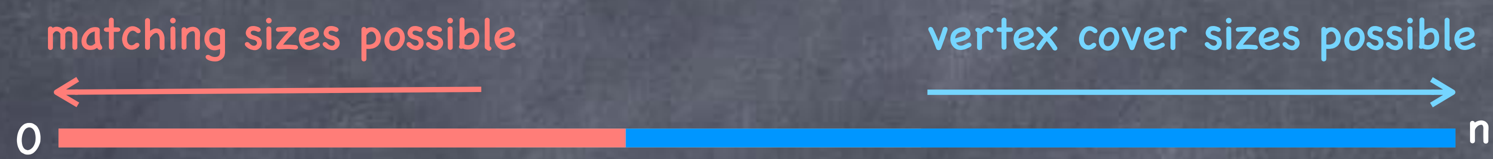
Graphs

Dilworth's Theorem



Min-Max Results



- Recall:
  - In a graph, size of any matching ≤ size of any vertex cover
    - In bipartite graphs, equality achieved! Kőnig's theorem
  - In a poset, size of any chain ≤ size of any anti-chain decomp
    - Equality is achieved! Mirsky's theorem
- Today:
  - In a poset, size of any anti-chain ≤ size of any chain decomp
    - Equality is achieved! Dilworth's theorem

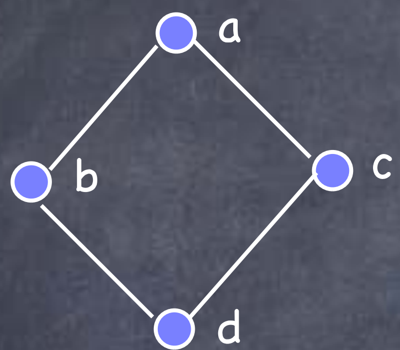
Each chain can have at most one element of an anti-chain.

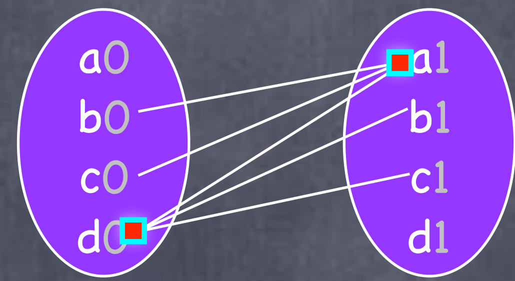
Dilworth's Theorem

- Dilworth's Theorem: The least number of chains needed to partition S is exactly the size of a largest anti-chain
- Easy direction: size of any anti-chain ≤ size of any chain decomp
- To prove: There is an anti-chain at least as large as a chain decomposition
  - Consider a poset $(S,\preccurlyeq)$, with $|S|=n$
  - Construct a bipartite graph G s.t.
    - a vertex cover of size ≤ t in G ⇒ antichain of size ≥ n-t
    - a matching of size ≥ t in G ⇒ partition S into ≤ n-t chains
  - Kőnig's theorem: there is a vertex cover and matching of the same size, say t, in G
  - Hence an antichain at least as large as a chain decomposition

Dilworth's Theorem

- Let $G = (S \times \{0\}, S \times \{1\}, E)$, where $E = \{ \{(u,0),(v,1)\} \mid u \preccurlyeq v, u \neq v \}$





$C=\{(d,0),(a,1)\}$
$B = \{a,d\}$
$A = \{b,c\}$

- Given vertex cover C, let $B = \{ u \mid \exists b \in \{0,1\}, (u,b) \in C \}$. Let $A=S-B$.
  - $|B| \leq |C| \Rightarrow |A| \geq |S|-|C|$
  - Also, A is an anti-chain
    [If $u,v \in A$, and $u \preccurlyeq v$, then $(u,0)$ and $(v,1) \notin C$, and edge $\{(u,0),(v,1)\} \in E$!]

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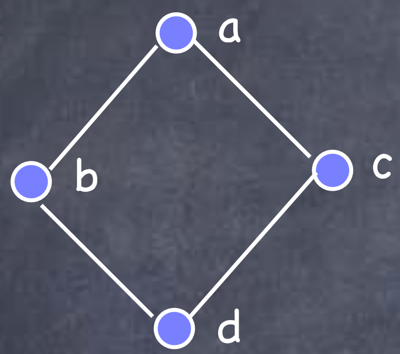

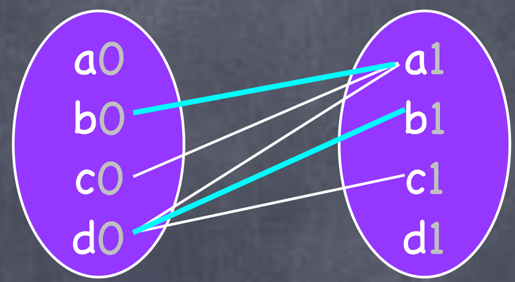

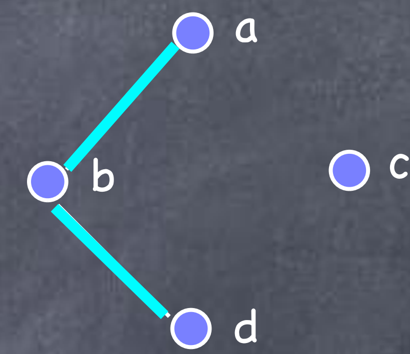

- Given a matching M, define a graph $F=(S,E^*)$, where $E^*=\{ \{u,v\} \mid \{(u,0),(v,1)\} \in M \}$.
  - F is a forest, with each connected component being a path
    - In F, any u will have degree ≤ 2 [one from (u,0), one from (u,1)]
    - F has no cycles [Cycle $v_0,v_1,\ldots,v_k \Rightarrow v_0 \preccurlyeq v_1 \preccurlyeq .. \preccurlyeq v_0$!]
  - Each such path in F forms a chain in the poset
  - Number of chains = number of connected components $= |S| - |E^*| = |S|-|M|$

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Comparison Graph

- Mirsky's theorem and Dilworth's theorem can be seen as statements about the comparison graph of the poset
- Given a poset $(S,\preccurlyeq)$, its comparison graph is $G=(S,E)$ where $E = \{ \{u,v\} \mid u\preccurlyeq v, u\neq v \}$
- If G is a comparison graph, any induced subgraph of G is also a comparison graph
- A chain corresponds to a clique in G, and an anti-chain corresponds to an independent set (i.e., a clique in $\overline{G}$)
- An anti-chain decomposition corresponds to a colouring of G, and a chain decomposition corresponds to a colouring of $\overline{G}$
- Mirsky's theorem: If G is a comparison graph, $\chi(G) = \omega(G)$
  Dilworth's theorem: If G is a comparison graph, $\chi(\overline{G}) = \omega(\overline{G})$

Comparison Graph

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- If G is a comparison graph, any induced subgraph of G is also a comparison graph
- **Perfect Graph:** G is a perfect graph if for every induced subgraph G' of G, $\chi(G') = \omega(G')$
- Mirsky: A comparison graph is perfect
  Dilworth: The complement of a comparison graph is perfect
- **Fact [Perfect Graph Theorem]**: G is perfect iff $\overline{G}$ is perfect
  - Note: Given Perfect Graph Theorem, Mirsky ⇔ Dilworth