Recursive Definitions

Generating Functions

Generating Functions

- A generating function is an alternate representation of an infinite sequence, which allows making useful deductions about the sequence (including, possibly, a closed form)
- Sequence f(0), f(1), ... is represented as the formal expression $G_f(X) \triangleq f(0) + f(1)\cdot X + f(2)\cdot X^2 + \ldots$ (ad infinitum)
  - i.e., for $f : \mathbb{N} \to \mathbb{R}$, we define $G_f(X) \triangleq \Sigma_{k\geq 0}\, f(k)\cdot X^k$
- e.g., If $f(k) = a^k$ for some $a \in \mathbb{R}$, $G_f(X) = \Sigma_{k\geq 0}\, a^k \cdot X^k$

"Ordinary Generating Functions"

Generating Functions

- Generating functions sometimes have a succinct representation
- e.g., For $f(k) = a^k$ for some $a \in \mathbb{R}$, $G_f(X) = \Sigma_{k \geq 0}\ a^k \cdot X^k$
  - If we substituted for X a real number x sufficiently close to 0, we have $|ax| < 1$ and this would converge to $1/(1-ax)$
  - So we write $G_f(X) = 1/(1-aX)$ (for sufficiently small $|X|$). This will later let us manipulate $G_f(X)$ algebraically

Extended Binomial Theorem

- A useful tool for manipulating/analysing generating functions
- For $a \in \mathbb{R}$, $\binom{a}{k} \triangleq \frac{a(a-1)\ldots(a-k+1)}{k!}$ $(k \in \mathbb{Z}^+)$, and $\binom{a}{0} \triangleq 1$
- Extended binomial theorem:

  For $|x|<1$, $a \in \mathbb{R}$, $(1+x)^a = \Sigma_{k\geq 0} \binom{a}{k} \cdot x^k$

- Useful in finding a closed form for f given G_f of certain forms
- e.g., $G_f(X) = 1/(1-X)$. Then, $\Sigma_{k\geq 0} f(k) \cdot X^k = (1-X)^{-1}$
  - $\binom{-1}{k} = (-1)(-2)\ldots(-k)/k! = (-1)^k \Rightarrow (1-X)^{-1} = \Sigma_{k\geq 0} X^k \Rightarrow f(k)=1$
- Similarly, $\binom{-2}{k} = (-2)(-3)\ldots(-k-1)/k! = (-1)^k(k+1)$

  $\Rightarrow$ $1/(1-X)^2 = \Sigma_{k\geq 0} (k+1) \cdot X^k$

Extended Binomial Theorem

- $G_f(X) = 1/(1-aX)^b$ for $f(k) = (-a)^k \cdot \binom{-b}{k} = \binom{b+k-1}{k} \cdot a^k$
  - e.g., b=1: $f(k) = a^k$. b=2: $f(k) = (k+1).a^k$
- $G_{f+g}(X) = G_f(X) + G_g(X)$
- $G_g(X) = X \cdot G_f(X)$, where $g(0)=0$ and $g(k+1) = f(k)$ for $k \geq 0$
- If a generating function G_f is known and has a nice form, then often using the extended binomial theorem, one can compute a closed-form expression for f
- But how do we get G_f ?

Generating Functions From Recurrence Relations

- e.g., $f(0)=0$, $f(1) = 1$. $f(n) = f(n-1) + f(n-2)$, $\forall n \geq 2$. [Fibonacci]
- $f(n) \cdot X^n = X \cdot f(n-1) \cdot X^{n-1} + X^2 \cdot f(n-2) \cdot X^{n-2}$ (for $n \geq 2$)

  $\Rightarrow \sum_{n\geq 2} f(n) \cdot X^n = X \cdot \sum_{n\geq 2} f(n-1) \cdot X^{n-1} + X^2 \cdot \sum_{n\geq 2} f(n-2) \cdot X^{n-2}$

  $\Rightarrow G_f(X) - f(0) - f(1) \cdot X = X \cdot (G_f(X) - f(0)) + X^2 \cdot G_f(X)$

  $\Rightarrow G_f(X)\,(1-X-X^2) = f(0) + (f(1)-f(0)) \cdot X$

  - $G_f(X) = X/(1-X-X^2)$
- More generally:
  $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2)$, $\forall n \geq 2$
  - $G_f(X) = (c + (d-ac)X)/(1-aX-bX^2)$

Generating Functions For Series Summation

- Suppose $g(k) = \Sigma_{j=0 \text{ to } k}\ f(j)$
- What is $G_g(X)$, in terms of $G_f(X)$?
  - Recursive definition: $g(0) = f(0)$. $g(n) = g(n-1) + f(n)$, $\forall n \geq 1$.
  - So, $\forall k \geq 1$, $g(k) \cdot X^k = g(k-1) \cdot X^{k-1} \cdot X + f(k) \cdot X^k$
  - $G_g(X) = g(0) + X \cdot G_g(X) + (G_f(X) - f(0))$
  - **$G_g(X) = G_f(X)/(1-X)$**