Recursive Definitions

Generating Functions

More Examples

Recall

Generating Functions

- For $f: \mathbb{N} \to \mathbb{R}$, we defined $G_f(X) \triangleq \Sigma_{k\geq 0} f(k) \cdot X^k$
- The extended binomial theorem
  - e.g., $G_f(X) = 1/(1-aX)^b$ for $f(k) = (-a)^k \cdot \binom{-b}{k}$
    $= \binom{b+k-1}{k} \cdot a^k$, for $b \in \mathbb{Z}^+$
- Combinations: e.g., $G_h(X) = G_f(X) + G_g(X)$, where $h(k)=f(k)+g(k)$
  $G_g(X) = \alpha X\, G_f(X)$, where $g(0) = 0$, $g(k) = \alpha f(k-1)$ $\forall k > 0$
  $G_h(X) = (1+\alpha X)\, G_f(X)$, where $h(0)=f(0)$, $h(k) = f(k) + \alpha f(k-1)$ $\forall k>0$
- From recurrence relations
  - e.g., If $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2)$, $\forall n \geq 2$
    - $G_f(X) = (c + (d-ac)X)/(1-aX-bX^2)$
  - e.g., If $g(k) = \Sigma_{j=0 \text{ to } k} f(j)$
    - $G_g(X) = G_f(X)/(1-X)$

Generating Functions For Series Summation

- e.g., $g(k) = \Sigma_{j=0 \text{ to } k} (j+1)^2$
- $G_g(X) = G_f(X)/(1-X)$ where $f(j) = (j+1)^2$
- Consider $G(X) = 1 + X + X^2 + \ldots = 1/(1-X)$
  - $G'(X) = 1 + 2 \cdot X + 3 \cdot X^2 + \ldots = 1/(1-X)^2$
  - Let $H(X) = X\,G(X) = X + 2 \cdot X^2 + 3 \cdot X^3 + \ldots = X/(1-X)^2$
  - So $H'(X) = 1 + 2^2 \cdot X + 3^2 \cdot X^2 + \ldots = 1/(1-X)^2 + 2X/(1-X)^3$
    $= (1+X)/(1-X)^3$

    is the generating function of $f(j) = (j+1)^2$.
- $G_g(X) = (1+X)/(1-X)^4$.
- Exercise: use ext. binomial theorem to compute coeff. of X^k

Calculus!

Alternately, from extended binomial theorem

Generating Functions For Counting Combinations

- e.g., Let $f(n)$ = number of ways to throw n unlabelled balls into d labelled bins (for some fixed number d)
  - Solution 1: Use stars and bars
  - Solution 2: Reason about $G_f(X)$
    - Coefficient of X^n in $G_f(X)$ must count the number of (non-negative integer) solutions of $n_1 + \ldots + n_d = n$
    - Can write $G_f(X) = (1+X+X^2+\ldots)^d$
    - So, $G_f(X) = [1/(1-X)]^d = (1-X)^{-d}$
    - Coefficient of X^n $= \binom{-d}{n}(-1)^n$

$= d\,(d+1) \ldots (d+n-1) / n! = C(d+n-1,n)$

Recall

A Closed Form

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2) \quad \forall n \geq 2$.
- Suppose $X^2 - aX - b = 0$ has two distinct (possibly complex) solutions, x and y
- Claim: $\exists p,q \; \forall n \; f(n) = p \cdot x^n + q \cdot y^n$
- Let $p=(d-cy)/(x-y)$, $q=(d-cx)/(y-x)$ so that base cases $n=0,1$ work
- Inductive step: for all $k \geq 2$
  Induction hypothesis: $\forall n$ s.t. $1 \leq n \leq k-1$, $f(n) = px^n + qy^n$
  To prove: $f(k) = px^k + qy^k$
  - $f(k) = a \cdot f(k-1) + b \cdot f(k-2)$
    $= a \cdot (px^{k-1}+qy^{k-1}) + b \cdot (px^{k-2}+qy^{k-2}) - px^k - qy^k + px^k + qy^k$
    $= - px^{k-2}(x^2-ax-b) - qy^{k-2}(y^2-ay-b) + px^k + qy^k = px^k + qy^k$ ✓

Recall

A Closed Form

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2)$ $\forall n \geq 2$.
- Suppose $X^2 - aX - b = 0$ has only one solution $x \neq 0$
  i.e., $X^2 - aX - b = (X-x)^2$, or equivalently, $a=2x$, $b=-x^2$
- Claim: **$\exists p,q\ \forall n\ f(n) = (p + q \cdot n)x^n$**
- Let $p = c$, $q = d/x-c$ so that base cases $n=0,1$ work
- Inductive step: for all $k \geq 2$
  Induction hypothesis: $\forall n$ s.t. $1 \leq n \leq k-1$, $f(n) = (p + qn)y^n$
  To prove: $f(k) = (p+qk)x^k$
  - $f(k) = a \cdot f(k-1) + b \cdot f(k-2)$
    $= a\ (p+qk-q)x^{k-1} + b \cdot (p+qk-2q)x^{k-2} - (p+qk)x^k + (p+qk)x^k$
    $= -(p+qk)x^{k-2}(x^2-ax-b) - qx^{k-2}(ax+2b) + (p+qk)x^k = (p+qk)x^k$ ✓

A Closed Form

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2)$ $\forall n \geq 2$.
- Recall: $G_f(X) = (c + (d-ac)X)/(1-aX-bX^2)$
- Let $G_f(X) = (\alpha + \beta X)/(1-aX-bX^2)$. i.e., $\alpha = c$, $\beta = d-ac$.
- Writing $Z = X^{-1}$, we have $G_f(X) = (\alpha Z^2 + \beta Z)/(Z^2-aZ-b)$
- Let $(Z^2-aZ-b) = (Z-x)(Z-y)$
  - $a = x+y$, $-b = xy$
  - $(1-aX-bX^2) = (1-xX)(1-yX)$
- Two cases: $x \neq y$ and $x = y$

A Closed Form

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2) \quad \forall n \geq 2$.
- $G_f(X) = (\alpha + \beta X)/[(1-xX)(1-yX),]$ where $\alpha = c$, $\beta = d-ac$, $a = x+y$, $-b = xy$.
- Case 1: $x \neq y$.
  - $1/[(1-xX)(1-yX)] = [\ 1/(1-xX) - 1/(1-yX)\ ]/[X(x-y)]$
  - Recall, $1/(1-xX) = \Sigma_{k\geq 0} (xX)^k$
  - So, $G_f(X) = (\alpha/X + \beta)/(x-y) \cdot \Sigma_{k\geq 0} (xX)^k - (yX)^k$
    $= \Sigma_{k\geq 1} \alpha(x\cdot(xX)^{k-1} - y\cdot(yX)^{k-1})/(x-y) + \Sigma_{k\geq 0} \beta((xX)^k-(yX)^k)/(x-y)$
    $= \Sigma_{k\geq 0} (px^k+qy^k)\cdot X^k$, where $p = (\alpha x+\beta)/(x-y)$, $q = (\alpha y+\beta)/(y-x)$
  - $f(n)$ = coefficient of $X^n = px^n + qy^n$
- $\alpha = c$, $\beta = d-ac = d-(x+y)c \Rightarrow p = (d-yc)/(x-y)$, $q = (d-xc)/(y-x)$,

A Closed Form

- $f(0) = c$. $f(1) = d$. $f(n) = a \cdot f(n-1) + b \cdot f(n-2) \quad \forall n \geq 2$.
- $G_f(X) = (\alpha + \beta X)/[(1-xX)(1-yX),]$ where $\alpha = c$, $\beta = d-ac$, $a = x+y$, $-b = xy$.
- Case 2: $x=y\neq 0$.
  - $G_f(X) = (\alpha + \beta X)/(1-xX)^2$
  - Recall, $1/(1-xX)^2 = \Sigma_{k\geq 0} (k+1).x^k \cdot X^k$
  - $(\alpha + \beta X)/(1-xX)^2 = \Sigma_{k\geq 0} (\alpha + \beta X)\cdot(k+1)\cdot x^k \cdot X^k$

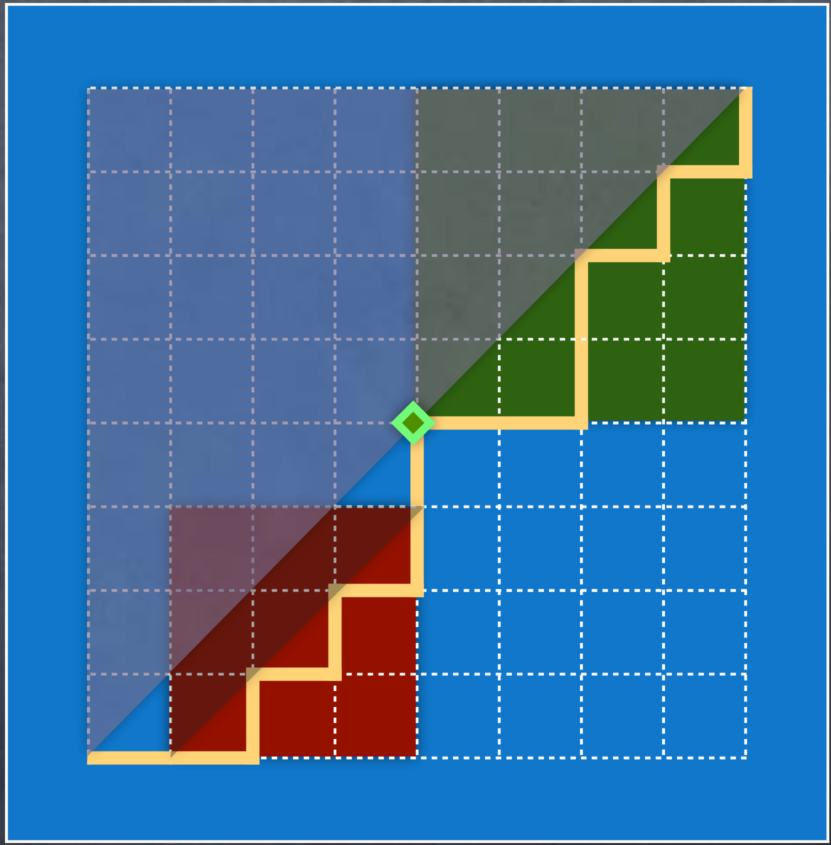
    $= \Sigma_{k\geq 0} (\alpha\cdot(k+1)\cdot x^k + \beta\cdot k \cdot x^{k-1})\cdot X^k$

    $= \Sigma_{k\geq 0} (p + qk)x^k \cdot X^k$, where $p = \alpha$, $q = (\alpha+\beta/x)$

Recall

Catalan Numbers

- How many paths are there in the grid from (0,0) to (n,n) without ever crossing over to the y>x region?
- Any path can be constructed as follows
  - Pick minimum k>0 s.t. (k,k) reached
  - (0,0) $\rightarrow$ (1,0) $\Rrightarrow$ (k,k-1) $\rightarrow$ (k,k) $\Rrightarrow$ (n,n) where $\Rrightarrow$ denotes a Catalan path
- $Cat(n) = \Sigma_{k=1 \text{ to } n}\ Cat(k-1) \cdot Cat(n-k)$
- Cat(0) = 1
- 1, 1, 2, 5, 14, 42, 132, ...



e.g., 42 = 1·14 + 1·5 + 2·2 + 5·1 + 14· 1

Closed form expression?

Catalan Numbers

- $Cat(n) = \Sigma_{k=1 \text{ to } n} Cat(k-1) \cdot Cat(n-k) \quad \forall n \geq 1$
- $Cat(n)\ X^n = \Sigma_{k=1 \text{ to } n} Cat(k-1) \cdot Cat(n-k) \cdot X^n$
  $= \text{term of } X^n \text{ in } X \cdot (\Sigma_{k \geq 1} Cat(k-1)\ X^{k-1}) \cdot (\Sigma_{k \leq n} Cat(n-k)\ X^{n-k}), \quad \forall n \geq 1$
  - For $n=0$, we have $Cat(0)\ X^0 = 1$
- $G_{Cat}(X) = 1 + X\ G_{Cat}(X)\ G_{Cat}(X)$
- Solving for G in $X \cdot G^2 - G + 1 = 0$, we have $G=[1 \pm \sqrt{(1-4X)}]/(2X)$
  - We need $\lim_{X \to 0} G_{cat}(X) = Cat(0) = 1$
    - $\lim_{X \to 0} [1 \pm \sqrt{(1-4X)}]/(2X) = \lim_{X \to 0} \pm (-4/[2\sqrt{(1-4X)}])/2 = \pm(-1)$

L'Hôpital's Rule

  - So we take $G_{cat}(X) = [1-\sqrt{(1-4X)}]/(2X)$
  - Then, what is the coefficient of X^n in $G_{cat}(X)$?

Catalan Numbers

- $G_{cat}(X) = [1-\sqrt{(1-4X)}]/(2X)$
- Then, what is the coefficient of X^k in $G_{cat}(X)$?
- Use extended binomial theorem:

$$(1-4X)^{1/2} = \Sigma_{k\geq 0} \binom{1/2}{k}(-4X)^k = 1 + \Sigma_{k\geq 1} -2\binom{2(k-1)}{k-1}/k \cdot X^k$$

- for k>0, $\binom{1/2}{k} = (1/2)(-1/2)\ (-3/2)\ (-5/2)\ \ldots\ (-(2k-3)/2)\ /k!$

$$= (-1)^{k-1}(1\cdot 1\cdot 3\cdot \ldots \cdot(2k-3))/[k!\ 2^k] = (-1)^{k-1}\binom{2k-2}{k-1}/[k\ 2^{2k-1}]$$

$2\cdot 4\cdot \ldots \cdot(2k-2)$

$(k-1)!\cdot 2^{k-1}$

- $G_{cat}(X) = \Sigma_{k\geq 1}\binom{2(k-1)}{k-1}/k \cdot X^{k-1}$
- $Cat(k)$ = Coefficient of X^k in $G_{cat}(X) = \binom{2k}{k}/(k+1)$