Countability and the Uncountable

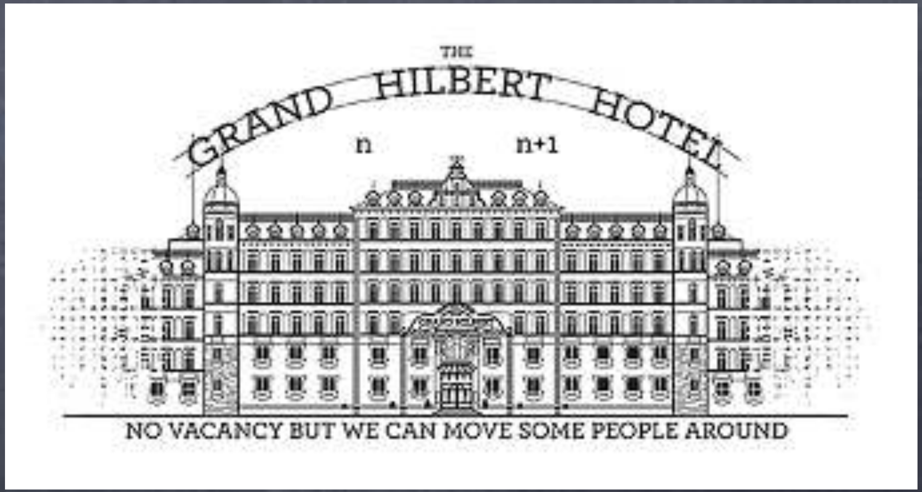


Image courtesy https://www.ias.edu/ideas/2016/pires-hilbert-hotel

Hilbert's Hotel

- The Grand Hilbert Hotel has infinite rooms indexed by $\mathbb{N}$

- The natural numbers are all staying in this hotel, with number n occupying Room no. n
- Suppose a new guest, -1 arrives
  - Can simply move everyone to the next room — n goes to Room no. n+1 — and make Room no. 0 available!
- Suppose all the negative integers arrive
  - No problem! Move n to Room no. 2n, and make all the odd numbered rooms available. Can send -n to 2n-1 now.
- But what if all the real numbers arrive?
  - The hotel can't accommodate them!

Image courtesy https://www.ias.edu/ideas/2016/pires-hilbert-hotel

How do you count infinity?

- How do you make precise the intuition that there are more real numbers than integers? Both are infinite...
- When do we say two infinite sets A & B have the same size?
- **Definition:** $|A| = |B|$ if there is a bijection from A to B

  Definition good for finite sets too

  - $|\mathbb{Z}| = |2\mathbb{Z}|$. ($2\mathbb{Z}$ = evens). $f : \mathbb{Z} \rightarrow 2\mathbb{Z}$ defined as $f(x)=2x$ is bijection
  - $|\mathbb{Z}| = |\mathbb{N}|$. bijection $g : \mathbb{Z} \rightarrow \mathbb{N}$: $g(x) = 2x$ for $x \geq 0$, $g(x)=2|x|-1$ for $x<0$
  - $|\mathbb{N}| = |2\mathbb{Z}|$. $h : \mathbb{N} \rightarrow 2\mathbb{Z}$ defined as $h = f \circ g^{-1}$

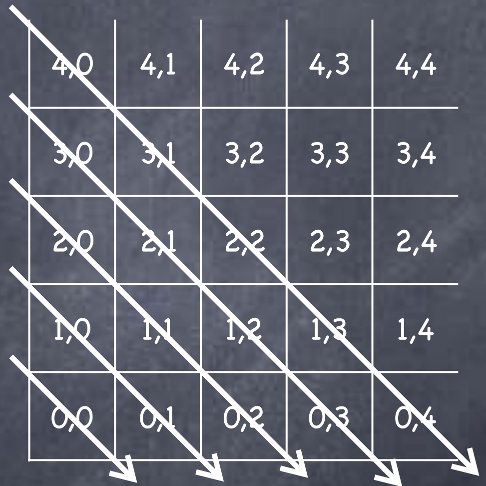
Countable

- A set A is **countably infinite** if $|A|=|\mathbb{N}|$
  - i.e., there is a bijection $f : \mathbb{N} \to A$
  - Note: $|A|=|\mathbb{N}|$ iff $|A|=|\mathbb{Z}|$, $|A|=|2\mathbb{Z}|$ etc.
- A set is **countable** if it is finite or countably infinite
- Intuition: all "discrete" sets are countable

How do you count infinity?

- **We defined:** A is countably infinite if $|A| = |\mathbb{N}|$, i.e., if there is a bijection between A and $\mathbb{N}$.
  - $\mathbb{N}^2$ is countable. Bijection by ordering points in $\mathbb{N}^2$ on a "curve"

    (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ...
    (i.e., f(0)=(0,0), f(1)=(1,0), f(2)=(0,1) ...)



  - Note: (0,0), (1,0), (2,0), (3,0) ... will not give a bijection
  - $\mathbb{Z}^2$ is countable. $f : \mathbb{Z}^2 \rightarrow \mathbb{N}$ defined as $f(a,b) = h(g(a),g(b))$, where $g : \mathbb{Z} \rightarrow \mathbb{N}$ and $h : \mathbb{N}^2 \rightarrow \mathbb{N}$ are bijections, is a bijection
  - More generally, if A and B are countable, the A×B is countable (extended to any finite number of sets by induction)

But Things Get Messy...

- Is $\mathbb{Q}$ countable?
  - We saw bijection between $\mathbb{Z}^2$ and $\mathbb{N}$. Enough to find a bijection between $\mathbb{Q}$ and $\mathbb{Z}^2$.
  - Not immediately clear: not all pairs (a,b) correspond to a distinct rational number a/b
    - a and b can have a common divisor; also, trouble with b=0
  - But easier to construct a one-to-one function $f : \mathbb{Q} \rightarrow \mathbb{Z}^2$ as $f(x) = (p,q)$ where $x=p/q$ is the "canonical representation" of x (i.e., $gcd(p,q)=1$ and $q > 0$).
    - Hence one-to-one function $g \circ f : \mathbb{Q} \rightarrow \mathbb{N}$, where $g : \mathbb{Z}^2 \rightarrow \mathbb{N}$ is a bijection

But Things Get Messy...

- Is $\mathbb{Q}$ countable?
  - One-to-one function $f : \mathbb{Q} \rightarrow \mathbb{N}$
  - Intuitively, if a one-to-one function from A to B, $|A| \leq |B|$
    - True for finite sets

Definition good for finite sets too

- **Definition:** $|A| \leq |B|$ if there is a one-to-one function from A to B
- So $|\mathbb{Q}| \leq |\mathbb{N}|$
- Also can construct a one-to-one function $h : \mathbb{N} \rightarrow \mathbb{Q}$ as $h(a)=a$. So $|\mathbb{N}| \leq |\mathbb{Q}|$
- Want to show $|\mathbb{Q}|=|\mathbb{N}|$ (i.e., a bijection between $\mathbb{Q}$ and $\mathbb{N}$)

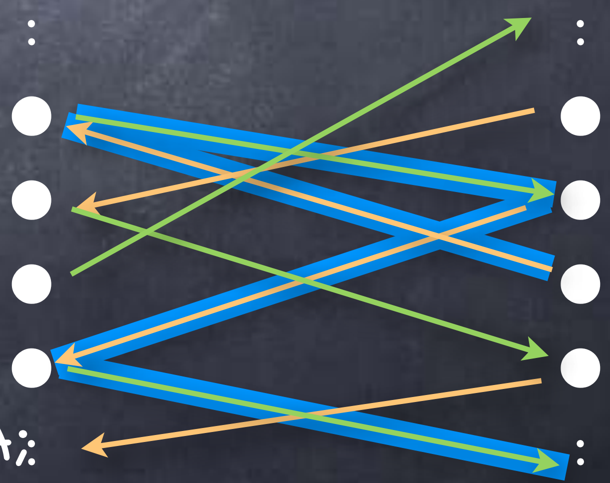
Bijection from Two Injections

Cantor-Schröder-Bernstein

- **Theorem [CSB]:** There is a bijection from A to B if and only if there is a one-to-one function from A to B, and a one-to-one function from B to A
- Restated: $|A|=|B| \Leftrightarrow |A| \leq |B|$ and $|B| \leq |A|$

Trivial for finite sets

- Proof idea: Let $f : A \to B$ and $g : B \to A$ (one-to-one).
- Consider infinite chains obtained by following the arrows
  - One-to-one ⇒ Each node in a unique chain
  - Chain could start from an A node, start from a B node or has no starting node (doubly infinite or cyclic). Say, types A,B and C
  - Let $h : A \to B$ s.t. $h(a)=f(a)$ if a's chain type A; else $h(a)=g^{-1}(a)$.



Find a perfect matching

Bijection from Two Injections

- Since $|\mathbb{Q}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Q}|$, by CSB-theorem $|\mathbb{Q}| = |\mathbb{N}|$
  - $\mathbb{Q}$ is countable
- Example: The set S of all finite-length strings made of [A-Z] is countably infinite
  - Interpret A to Z as the non-zero digits in base 27. Given $s \in S$, interpret it as a number. This mapping ($S \rightarrow \mathbb{N}$) is one-to-one (because no leading zeroes).
  - Map an integer n to A^n (string with n As). This is one-to-one.

Bijection from Two Injections

- Another example: $\mathbb{T}$ be the set of all infinitely long binary strings
- Claim: $|\mathbb{T}| = |\mathbb{R}|$
  - $|\mathbb{T}| \leq |\mathbb{R}|$: treat a binary string $s_1s_2s_3\ldots$ as the real number $0.s_1s_2s_3\ldots$ <u>in decimal</u>
    - This is a one-to-one mapping: a finite difference between the real numbers that two different strings map to
  - $|\mathbb{R}| \leq |\mathbb{T}|$: map each $x \in \mathbb{R}$ injectively to a real number in (0,1), say as $(1+e^{-x})^{-1}$ and then injectively to the string in its binary expansion
- Corollary: $|\mathbb{R}^2| = |\mathbb{R}|$
  - Because $|\mathbb{T}^2|=|\mathbb{T}|$ (bijection by interleaving), and $|\mathbb{R}^2|=|\mathbb{T}^2|$

Not true if we used binary representation instead of decimal representation, we'll have strings 011111.. and 10000... map to the same real number

Summary

- **Definition:** $|A| = |B|$ if there is a bijection from A to B
- **Definition:** $|A| \leq |B|$ if there is a one-to-one function from A to B
- **Theorem [CSB]:** $|A|=|B| \Leftrightarrow |A| \leq |B|$ and $|B| \leq |A|$
- A is **countably infinite** if $|A|=|\mathbb{N}|$
  - e.g., $|\mathbb{Z}|=|\mathbb{N}|$, $|2\mathbb{Z}|=|\mathbb{N}|$, $|\mathbb{N}^2|=|\mathbb{N}|$ etc. (saw explicit bijections)
  - e.g., $|\mathbb{Q}|=|\mathbb{N}|$ (saw one-to-one functions in both directions)
- A is **uncountable** if A is infinite but not countably infinite
  - Equivalently, if no function $f : A \to \mathbb{N}$ is one-to-one
  - Equivalently, if no function $f : \mathbb{N} \to A$ is onto

Uncountable Sets

- Claim: ℝ is uncountable
- Related claims:
  - Set 𝕋 of all infinitely long binary strings is uncountable
    - Contrast with set of all finitely long binary strings, which is a countably infinite set
  - The power-set of ℕ, ℙ(ℕ) is uncountable
    - There is a bijection $f : \mathbb{T} \rightarrow \mathbb{P}(\mathbb{N})$ defined as $f(s) = \{ i \mid s_i = 1 \}$
- How do we show something is not countable?!
- Cantor's "diagonal slash"

e.g., set of even numbers corresponds to the string 101010...

Cantor's Diagonal Slash

- To prove $\mathbb{P}(\mathbb{N})$ is uncountable
- Take any function $f : \mathbb{N} \rightarrow \mathbb{P}(\mathbb{N})$
- Make a binary table with $T_{ij} = 1$ iff $j \in f(i)$
- Consider the set $X \subseteq \mathbb{N}$ corresponding to the "flipped diagonal"
  - $X = \{ j \in \mathbb{N} \mid T_{jj} = 0 \}$
    $= \{ j \in \mathbb{N} \mid j \notin f(j) \}$
- X doesn't appear as a row in this table (why?)
  - So f not onto

	0	1	0	0	1	1	1
f(0) =	1	0	0	1	0	0	0
f(1) =	0	0	1	0	1	0	0
f(2) =	1	1	1	1	1	1	1
f(3) =	1	1	0	1	0	1	0
f(4) =	1	1	0	0	0	0	1
f(5) =	0	1	0	1	1	0	1
f(6) =	0	1	0	1	0	1	0

Cantor's Diagonal Slash

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Generalizes:

No onto function $f : A \rightarrow \mathbb{P}(A)$

for any set A

May not have a table enumerating f

(if A is uncountable)

Let $X = \{ j \in A \mid j \notin f(j) \}$

Claim: $\nexists\, i \in A$ s.t. $X = f(i)$

Suppose not: i.e., $\exists i, X = f(i)$.

$i \in X \leftrightarrow i \in f(i) \leftrightarrow i \notin X$

Contradiction!

Reals are Uncountable

- We saw $\mathbb{P}(\mathbb{N})$ is uncountable, i.e., $|\mathbb{P}(\mathbb{N})| \neq |\mathbb{N}|$
- Recall $\mathbb{T}$, the set of infinite binary strings
- We saw $|\mathbb{T}| = |\mathbb{P}(\mathbb{N})|$ (via an easy bijection) and $|\mathbb{T}| = |\mathbb{R}|$ (via CSB)
- Hence $|\mathbb{R}| \neq |\mathbb{N}|$, i.e., $\mathbb{R}$ uncountable
- $|\mathbb{R}|$ is a "higher infinity" than $|\mathbb{N}|$
  - $|\mathbb{P}(\mathbb{R})|$ even higher

Denoted as $\aleph_0, \aleph_1, \aleph_2, \ldots$