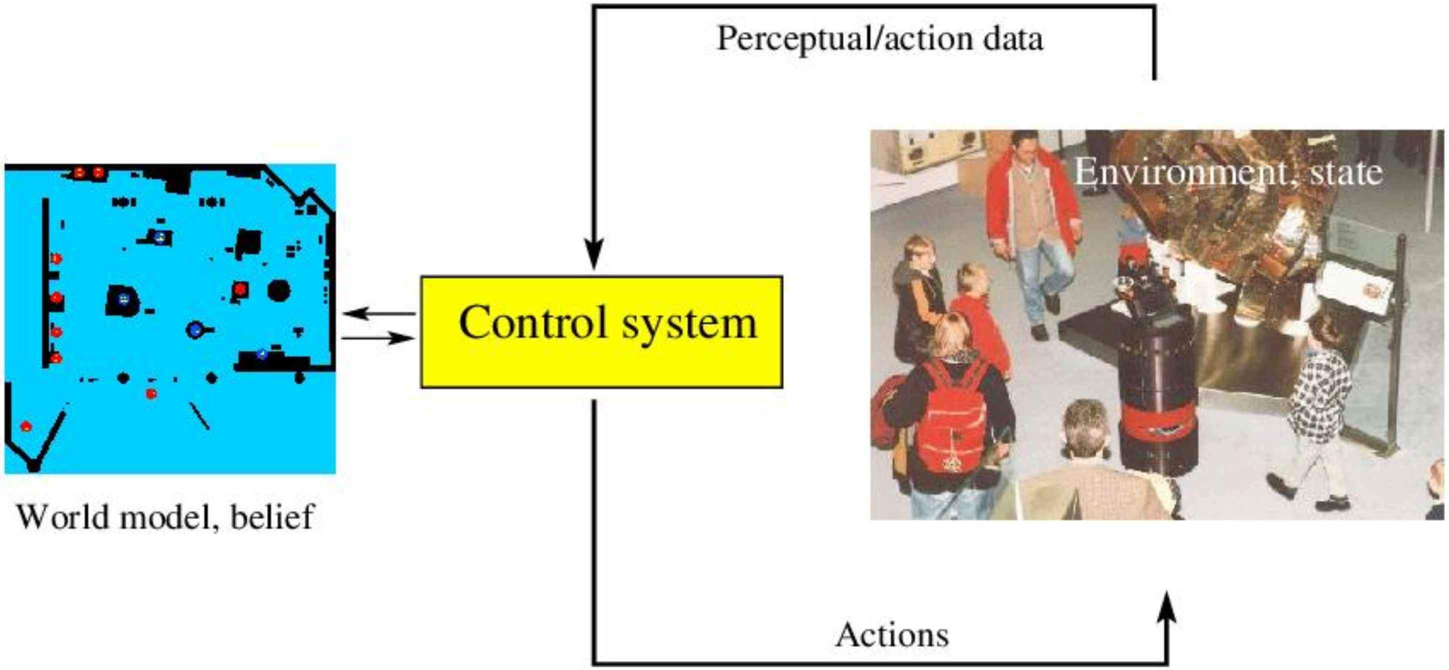
Recursive State Estimation

Nived Chebrolu

Slides have been adapted from Cyrill Stachniss at University of Bonn

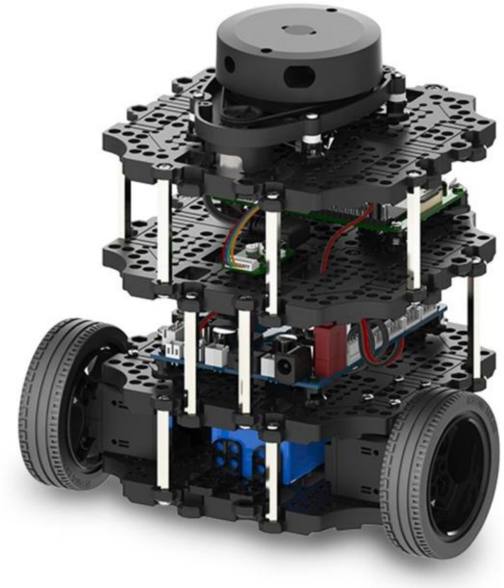
Robot Environment Interaction



Courtesy: Thrun, Burgard and Fox

State

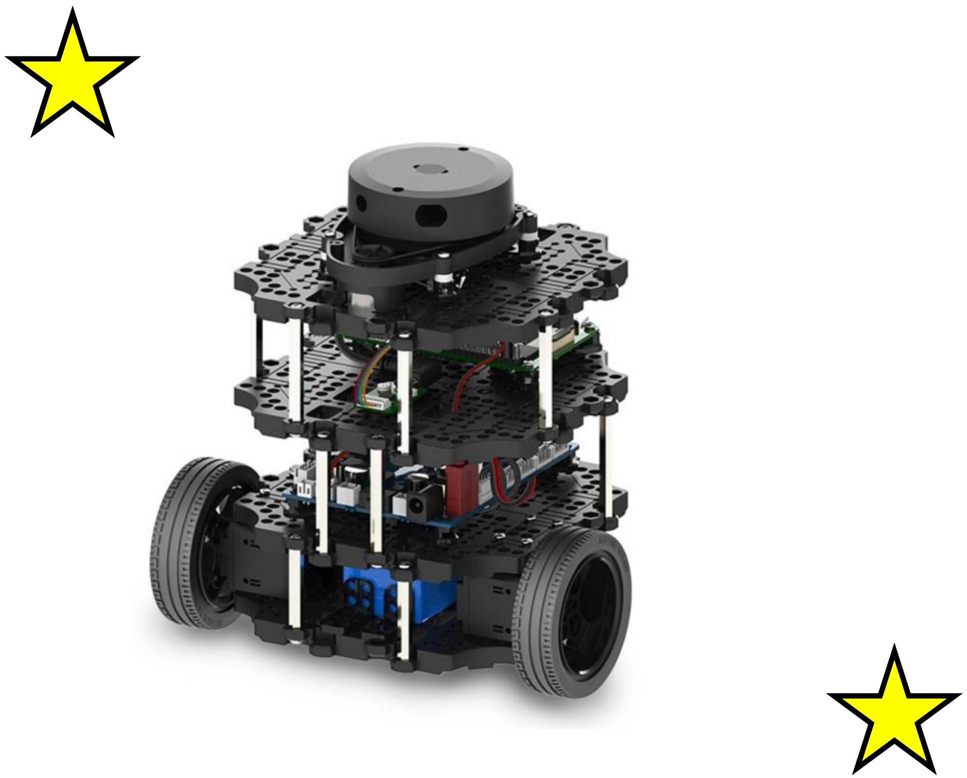
- state x



$$x = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

State

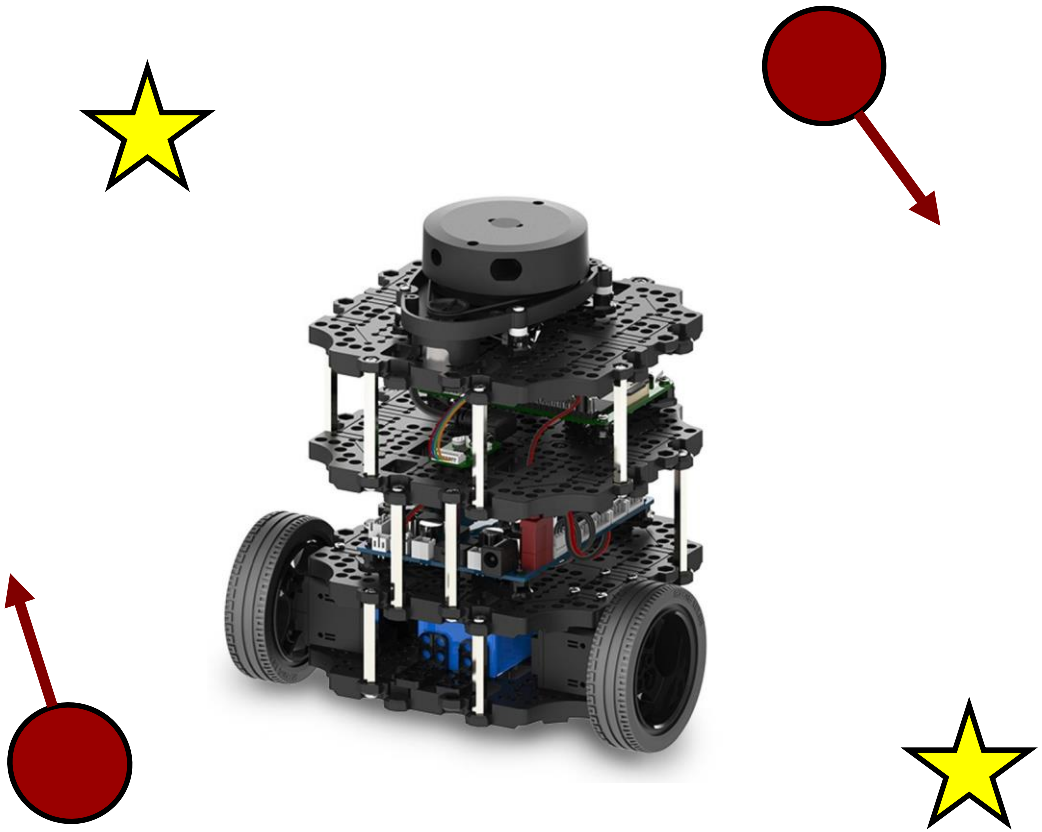
- state x



$$x = \begin{bmatrix} x \\ y \\ \theta \\ l_1 \\ l_2 \end{bmatrix}$$

State

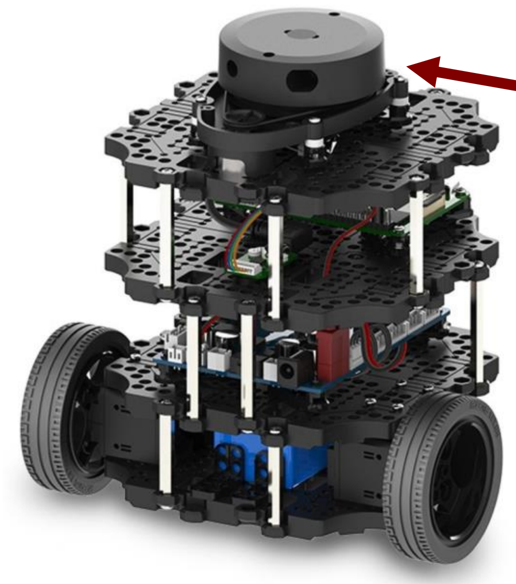
- state x

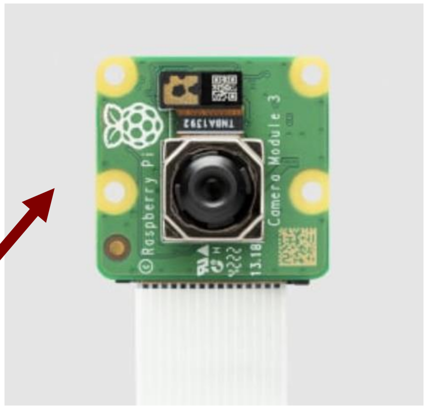


$$x = \begin{bmatrix} x \\ y \\ \theta \\ l_1 \\ l_2 \\ d_1 \\ d_2 \end{bmatrix}$$

Sensor Measurements

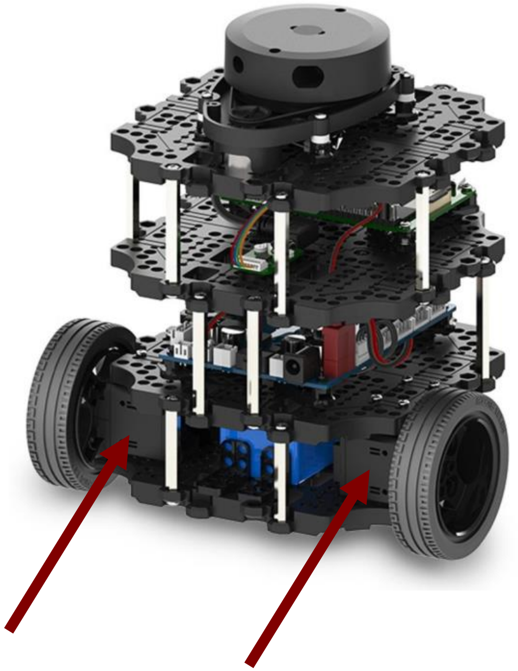
- measurements or observation z





Control data

- controls u



wheel encoders

Source: http://www.active-robots.com/

State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x \mid z, u)$$

State Estimation

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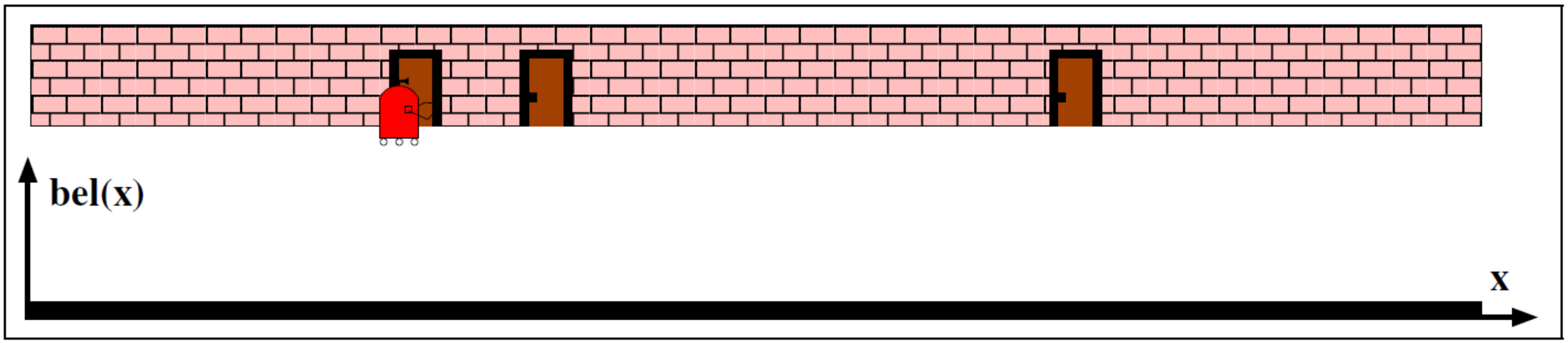
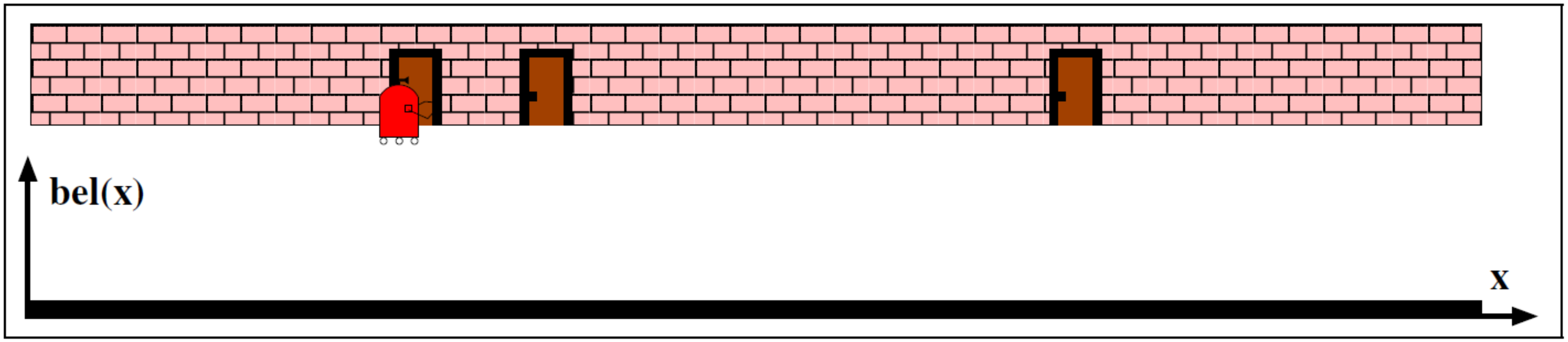
$$p(x_t | z_{1:t}, u_{1:t})$$

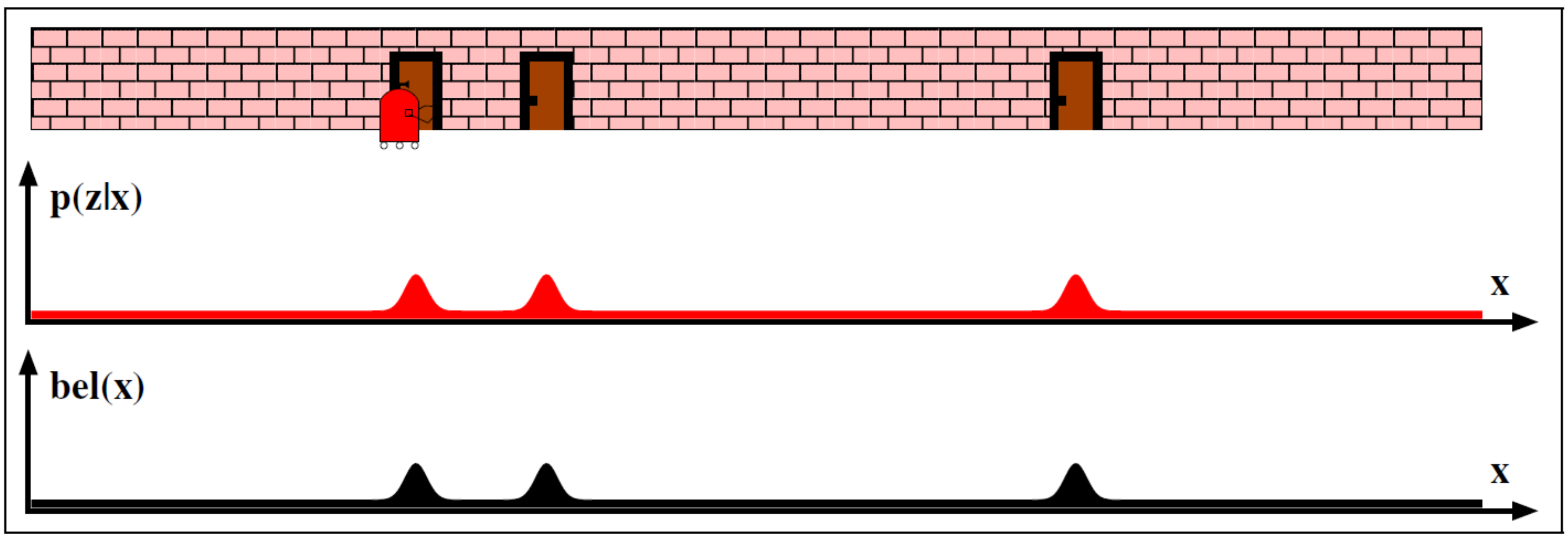
Recursive Bayes Filter



Courtesy: Thrun, Burgard and Fox

Recursive Bayes Filter





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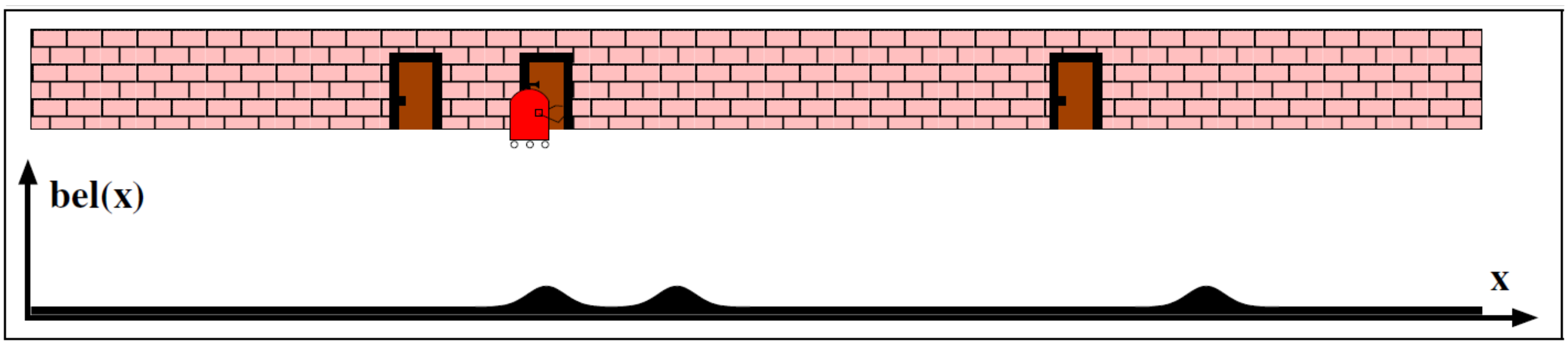
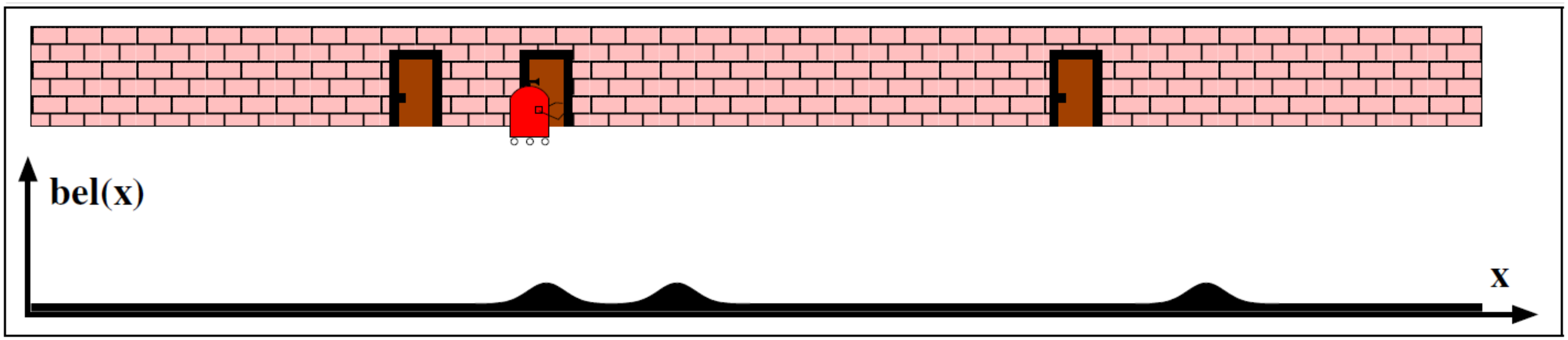
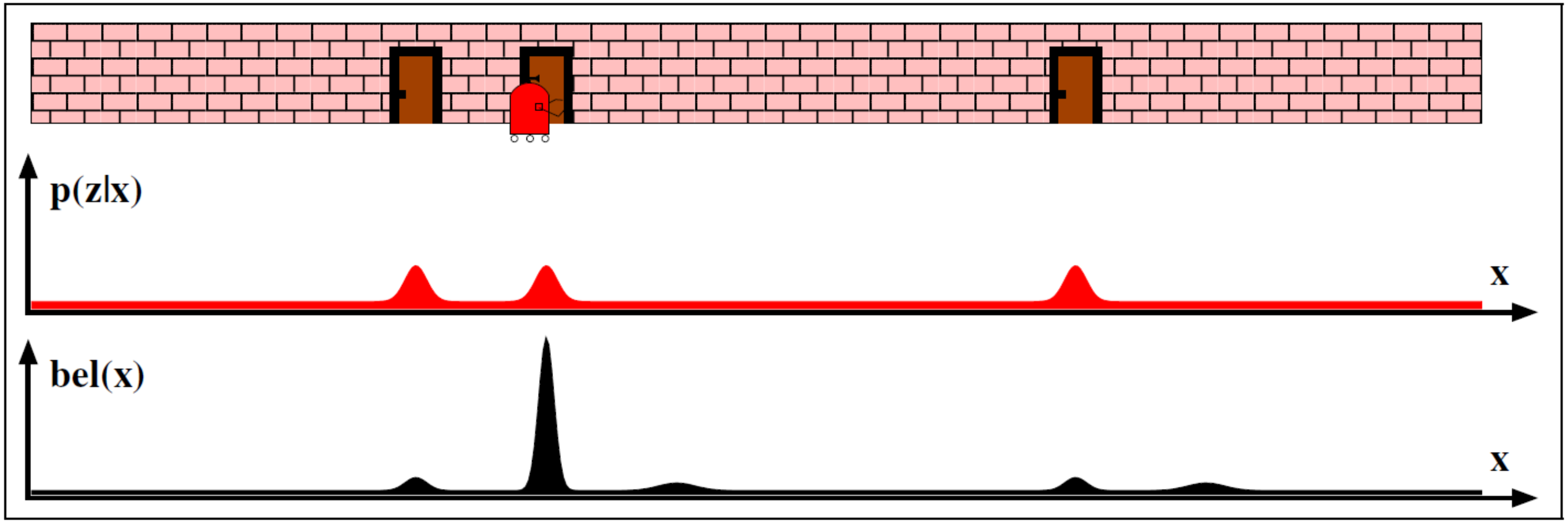
Recursive Bayes Filter



Courtesy: Thrun, Burgard and Fox

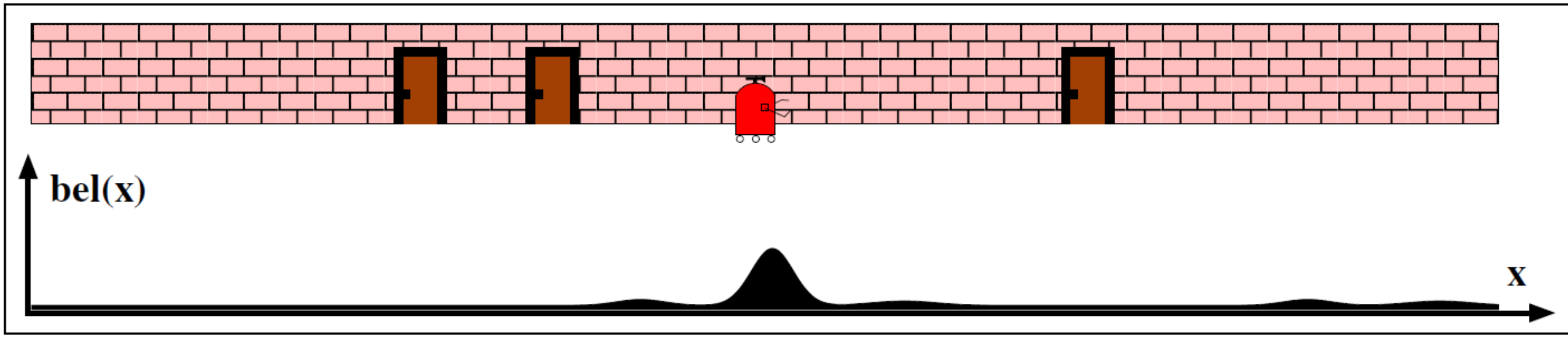
Recursive Bayes Filter





Courtesy: Thrun, Burgard and Fox

Recursive Bayes Filter



Courtesy: Thrun, Burgard and Fox

State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x_t | z_{1:t}, u_{1:t})$$

Recursive Bayes Filter 1

$$bel(x_t) = \underline{p(x_t \mid z_{1:t}, u_{1:t})}$$

definition of the belief

Recursive Bayes Filter 2

$$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$
$$= \underline{\eta \, p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \, p(x_t \mid z_{1:t-1}, u_{1:t})}$$

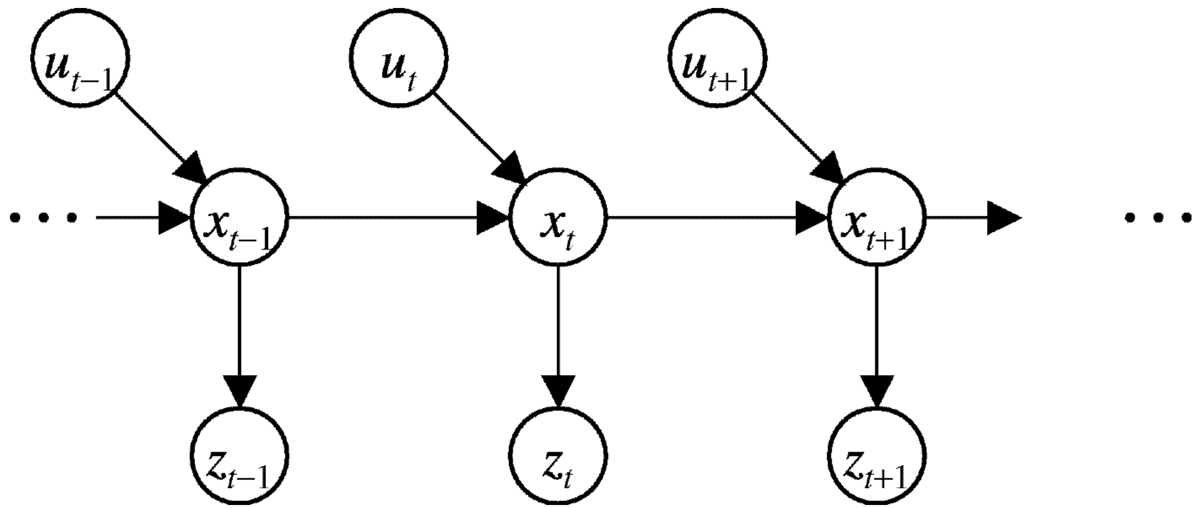
Bayes' rule

Recursive Bayes Filter 3

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta \, \underline{p(z_t \mid x_t)} \, p(x_t \mid z_{1:t-1}, u_{1:t})
\end{aligned}
$$

Markov assumption

Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t \mid x_t)$$

19

Recursive Bayes Filter 4

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \underline{\int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})} \\
&\qquad \underline{p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}}
\end{aligned}
$$

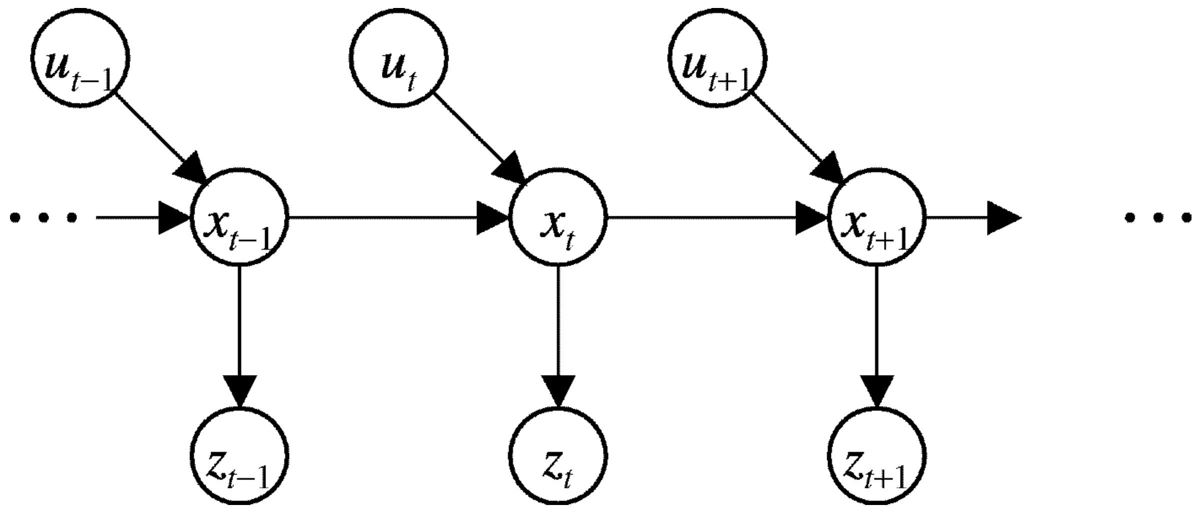
Law of total probability

Recursive Bayes Filter 5

$$
\begin{aligned}
bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t) \, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1} \\
&= \eta \, p(z_t \mid x_t) \int \underline{p(x_t \mid x_{t-1}, u_t)} \, p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1}
\end{aligned}
$$

Markov assumption

Markov Assumption



$$p(x_t \mid x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Recursive Bayes Filter 6

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:\underline{t-1}})\, dx_{t-1}
\end{aligned}
$$

independence assumption

Recursive Bayes Filter 7

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, \underline{bel(x_{t-1})}\, dx_{t-1}
\end{aligned}
$$

recursive term

Complete Derivation of the Recursive Bayes Filter

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t) \, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta \, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1} \\
&= \eta \, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \, p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1} \\
&= \eta \, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) \, dx_{t-1} \\
&= \eta \, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \, bel(x_{t-1}) \, dx_{t-1}
\end{aligned}
$$

Prediction and Correction Step

- Bayes filter can be written as a two step process

$$bel(x_t) = \underline{\eta\, p(z_t \mid x_t)}\ \underline{\int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\, dx_{t-1}}$$

- **Prediction step**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\, dx_{t-1}$$

- **Correction step**

$$bel(x_t) = \eta\, p(z_t \mid x_t)\ \overline{bel}(x_t)$$

Motion and Observation Model

- Prediction step

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{\textbf{motion model}} \, bel(x_{t-1}) \, dx_{t-1}$$

- Correction step

$$bel(x_t) = \eta \, \underbrace{p(z_t \mid x_t)}_{\textbf{observation model}} \, \overline{bel}(x_t)$$

(also: measurement or sensor model)

Different Realizations

- The Bayes filter is a **framework** for recursive state estimation
- There are **different realizations**
- **Different properties**
  - Linear vs. non-linear models for motion and observation models
  - Gaussian distributions only?
  - Parametric vs. non-parametric filters
  - …

Popular Filters

- **Kalman filter & EKF**
  - Gaussians
  - Linear or linearized models

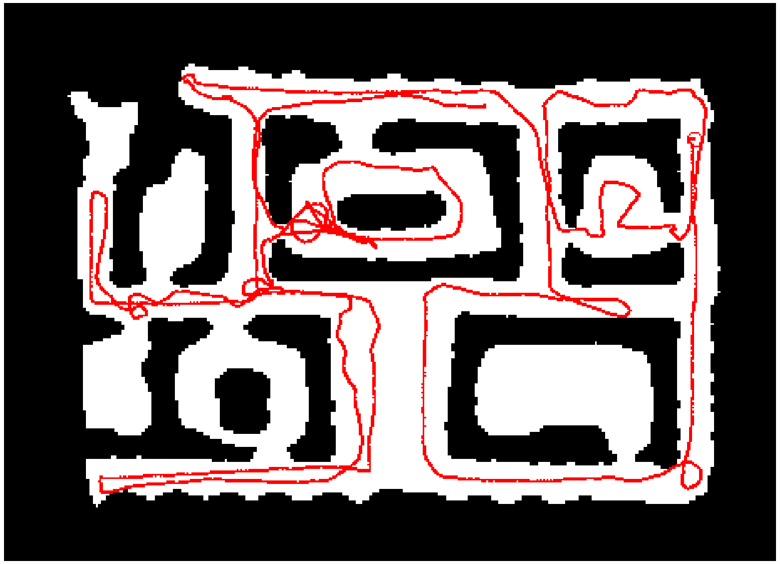
- **Particle filter**
  - Non-parametric
  - Arbitrary models (sampling required)

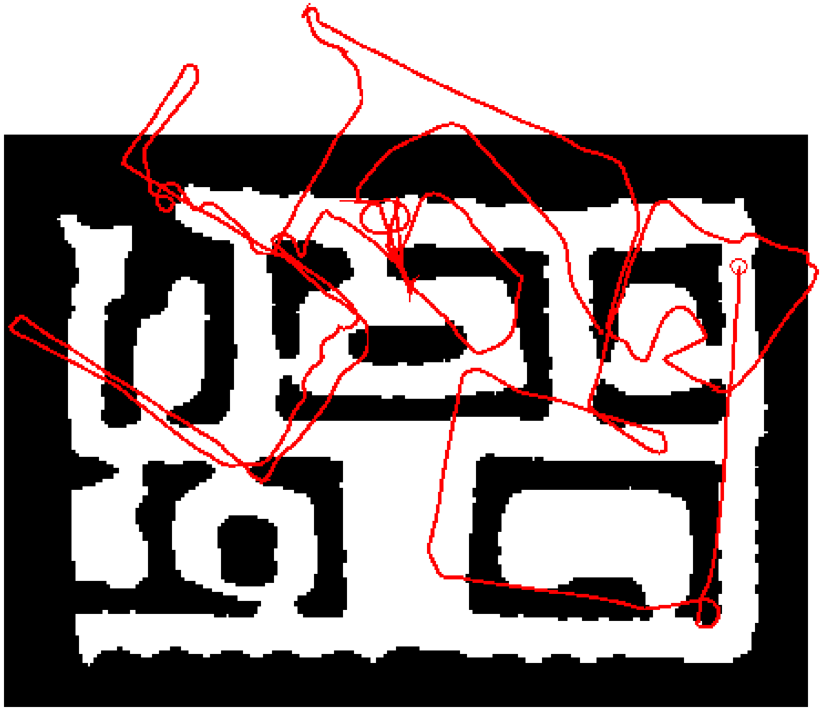
Motion Model

$$\overline{bel}(x_t) = \int \boxed{p(x_t \mid u_t, x_{t-1})}\, bel(x_{t-1})\, dx_{t-1}$$

Basic Motion Models

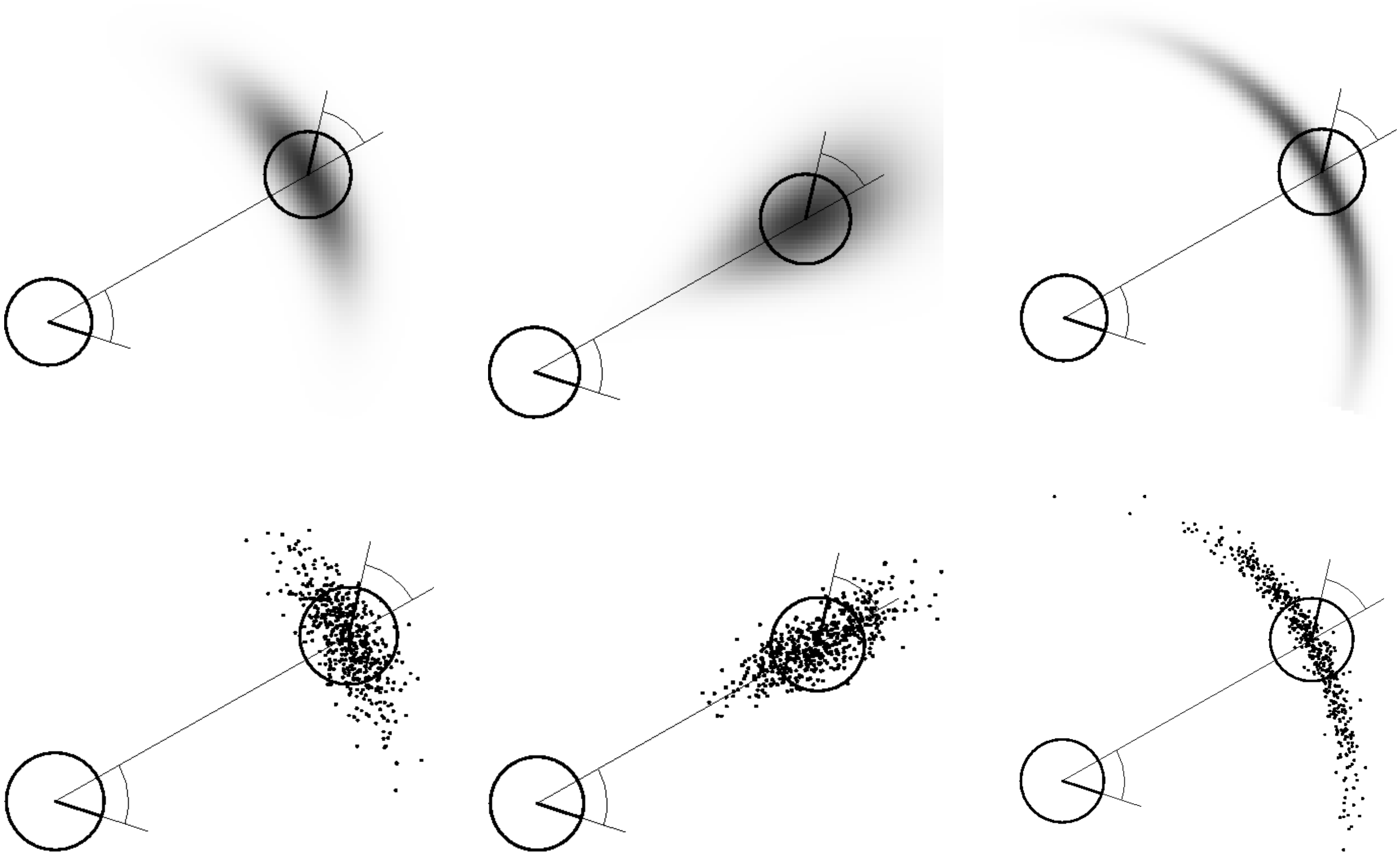
- Motion is inherently uncertain
- How can we model this uncertainty?





Courtesy: Thrun, Burgard, Fox

Example: Odometry-Based Motion



Courtesy: Thrun, Burgard, Fox

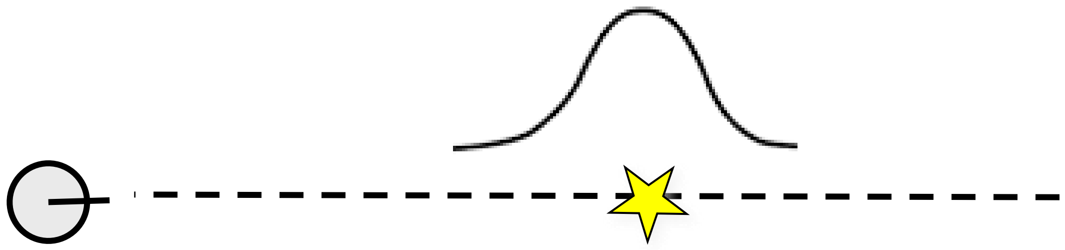
Observation Model

$$bel(x_t) = \eta \boxed{p(z_t \mid x_t)} \overline{bel}(x_{t-1})$$

Range Sensors

Example: Simple Observation Model with Gaussian Noise

- Range sensor estimating the distance to the closest obstacle
- Gaussian noise in the range reading



Model for Perceiving Landmarks with Range-Bearing Sensors

- Range-bearing $z_t^i = (r_t^i, \phi_t^i)^T$
- Pose $(x, y, \theta)^T$
- Observation of feature j at location $(m_{j,x}, m_{j,y})^T$

$$\begin{pmatrix} r_t^i \\ \phi_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{pmatrix} + Q_t$$

What if monocular cameras are used?

Summary

- Probabilities occur is most of the problems addressed here
- Bayes filter is a framework for state estimation
- There are different realizations of the Bayes filter that we will study in this course (e.g., EKF, particle filter)
- Motion and observation model are central models in the Bayes filter to be specified

Literature

Probability Primer

- Thrun et al. "Probabilistic Robotics", Chapter 2.1 & 2.2

On the Bayes filter

- Thrun et al. "Probabilistic Robotics", Chapter 2.3